Oide

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Supporting the Professional Learning of School Leaders and Teachers

National Seminar 9

Applied Mathematics

Welcome & Introductions

Introducing Oide

Oide

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Supporting the Professional Learning of School Leaders and Teachers

Lárionad Ceannaireachta Scoile
CSL
Centre for School Leadership

An tSraith Shóisearach do Mhúinteoirí
Junior CYCLE for teachers

NIPT
An Clár Náisiúnta Ionduchtaithe do Mhúinteoirí
The National Induction Programme for Teachers

PDST
Professional Development Service for Teachers | An tSeirbhís um Fhorbairt Ghairmiúil do Mhúinteoirí

Applied Maths Team

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Schedule

09:30 - 11:00	Taking Stock of The Journey So Far Supporting Students with The Modelling Project
11:00 - 11:15	Tea and Coffee
11:15 - 13:00	Modelling with Multi-Stage Dynamic Programming
13:00 - 14:00	Lunch
14:00 - 15:30	Exploring Difference Equations

Key Messages

Core to the specification is a non-linear approach which will promote the making of connections between various learning outcomes.

Strand 1 is the unifying strand and emphasises the importance of utilising mathematical modelling across all learning outcomes.

Applied Mathematics is rooted in authentic problems as a context for learning about the application of Mathematics.

Professional Development Supports

Overview of Support to Date

- 8 National Seminars
- 4 Collaboratives
- 2 Technology Workshops

Slides and additional Resources available

- 4 Webinars
- Video resources

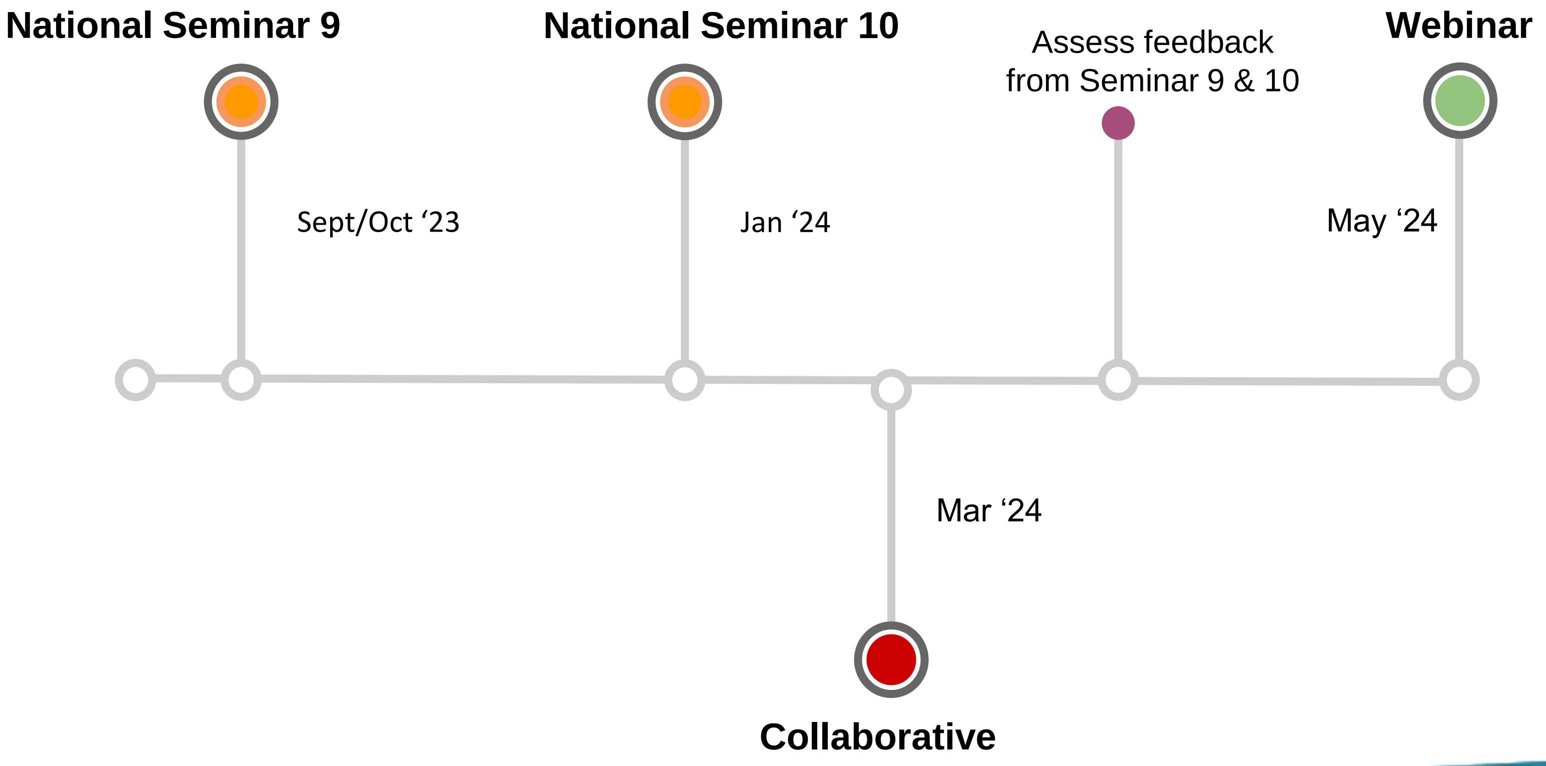
Recordings available online

https://www.pdst.ie/post-primary/sc/appliedmaths/cpd-resources (Oide link will also be provided)

Professional Development Supports

Overview of Upcoming Support

Year 4 September 2023 - May 2024



By The End of This Session You Will Have:

Discussed your experience of teaching the specification and your learning to date.

Engaged with the formulating stage of a modelling problem and having worked with others see how students' key skills that can be developed through pedagogy in the classroom.

Explored the possible ways of supporting students and developing their modelling skills before, during and after the modelling project.

Taking Stock

Engagement with the Specification

Now that the first two-year cycle of teaching the specification has been completed,

- What are your key takeaways?
- How can you best use mathematical modelling methods in your classroom?

Mentimeter

Oide

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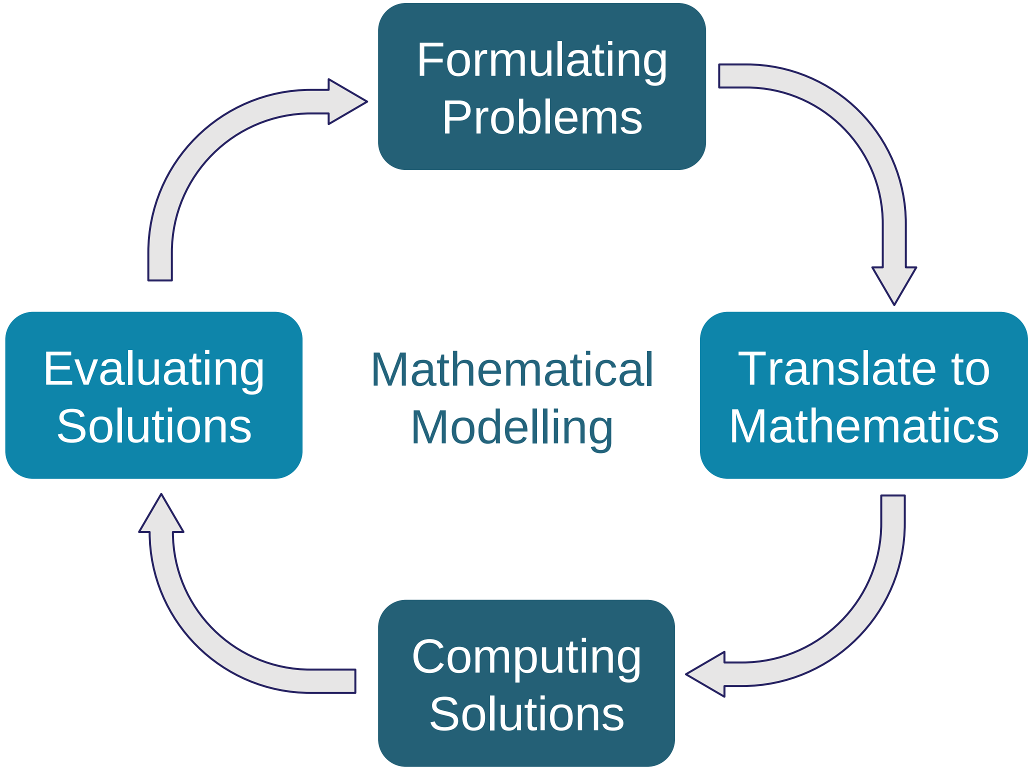
Supporting the Professional Learning of School Leaders and Teachers

Supporting Students with the Modelling Project

Taking Stock

The Modelling Project

Having supported students in completing a modelling project, what **nuggets of wisdom** would you give a teacher who is engaging with it for the first time this year?



Mentimeter

The Modelling Cycle
Supporting Students

How best can teachers support students,

before…

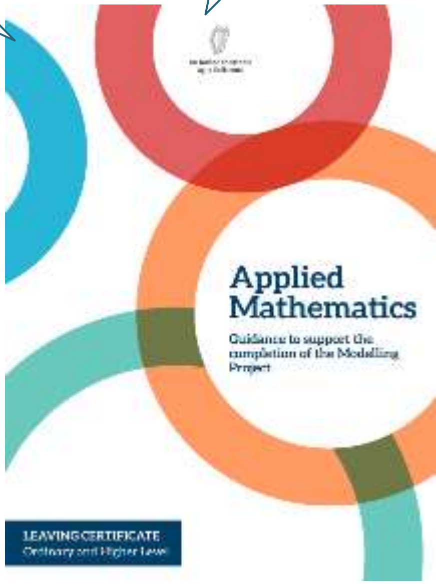
during…

after…

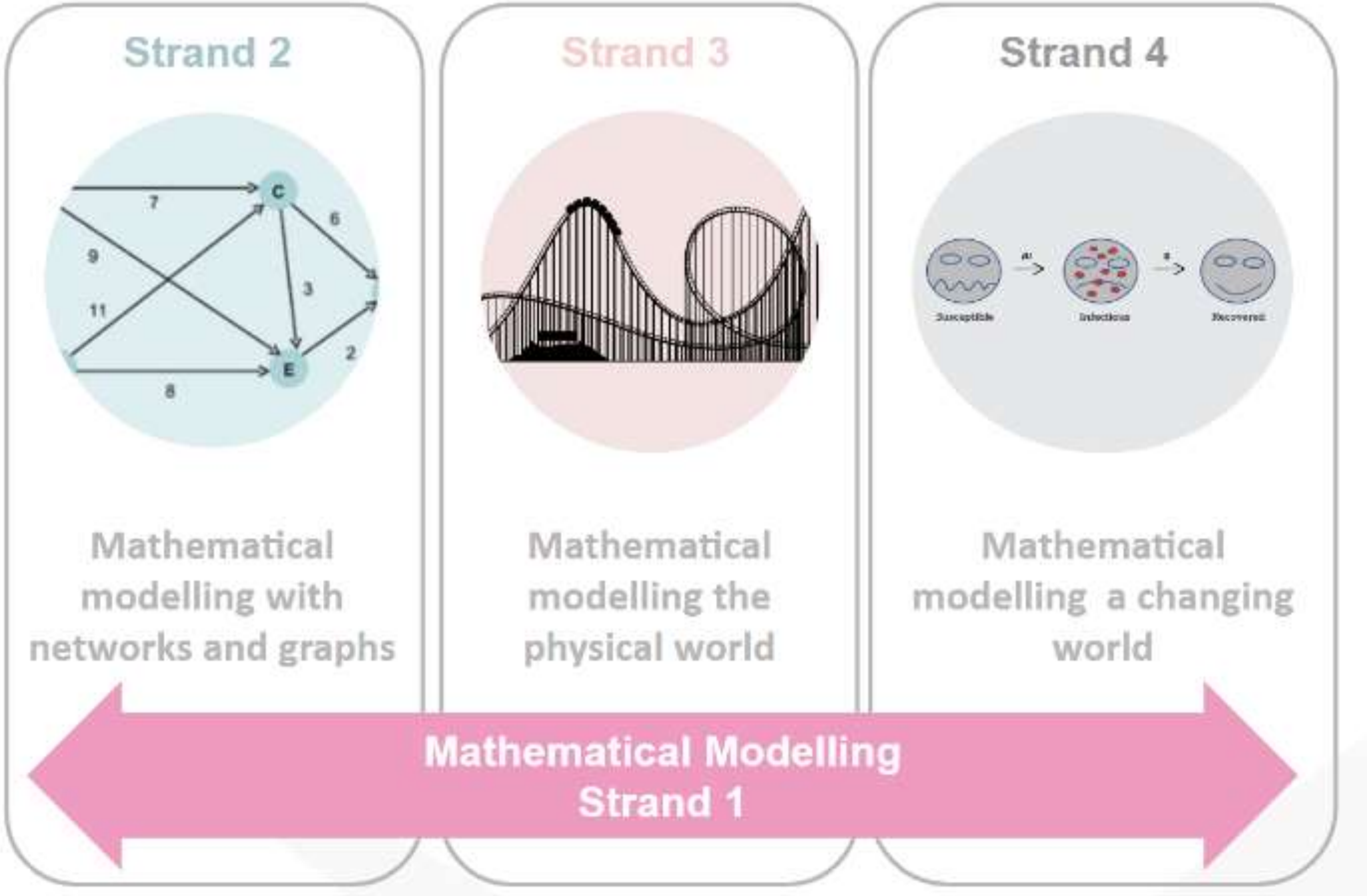
the modelling project?

“Prompting the student’s critical thinking in relation to the theme set out in the brief.”

“Providing instructions at strategic intervals to facilitate the timely completion of the modelling project.”



The Modelling Cycle
Supporting Students



L.C. Applied Maths Mathematical Modelling: Self-Assessment Tool

Formulating the Problem:

What is the problem being asked and what research must you do?

What variables (factors) will affect your model and what assumptions will you make?

Can you predict what the output of your model will achieve and for what context (who/what) will be affected by your model?

Translating the Problem to Mathematics:

What mathematical approach will you use to solve the problem and why?

Where will your assumptions and variables be used in your model?

It is fine for a problem to have more than one solution to it depending on the assumptions chosen.

Computing the Solution:

How did you calculate your solution and what affect did your variables and assumptions have on it?

What tools (technology etc.) did you use in your solution and did this enhance your calculations?

How will you present your solution (graphs, charts, other visual aids)?

Evaluating the Solution:

How accurate is your solution based on your earlier assumptions?

Can you refine/alter your assumptions to improve your solution and will this change your solution much?

Next iteration

It is important to show how your model is improving with each iteration and why you altered your assumptions/approach.

Presenting your Final Model:

How will you present your final model so that it is well presented and easy to read?

Can visual aids be used to better communicate your work?

It may be helpful to present your work so that someone unfamiliar to your project will understand it.

The Modelling Cycle
Formulating Problems

Consider the following context:

The 2024 European football Championship takes place at multiple venues across Germany in June/July. A key feature of a team's preparation for this is planning the logistics of travel, accommodation, purchasing and allocating stock for the team and scheduling a team's itinerary.

Select one or more aspects of logistical planning and model the problem(s) using *The Modelling Cycle*.

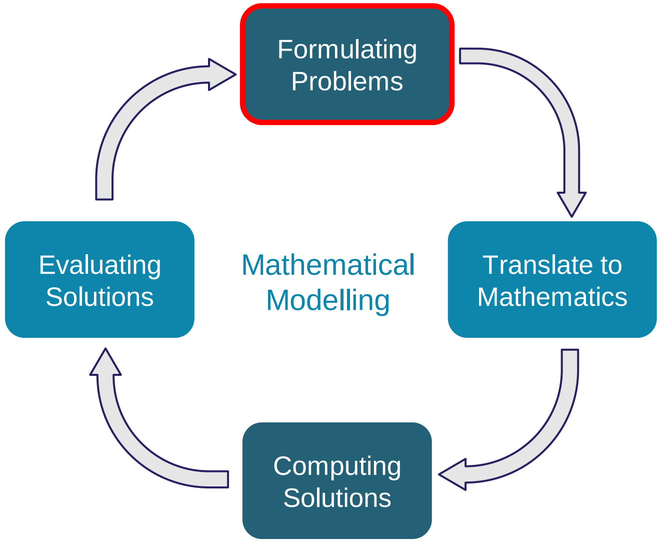
The Modelling Cycle
Formulating Problems

"determine what assumptions are necessary to simplify the problem situation"
Specification p.16

What problem statement could students initially choose to investigate?

What research and assumptions would be required for students?

What is your problem statement and what research must you do?
What variables (factors) are relevant to the problem?
Can you simplify the problem into smaller manageable parts?
Consider if there are limitations to your model due to your chosen assumptions?
Can you predict what the output of your model will achieve?



The Modelling Cycle
Supporting Students

Candidates should ensure that they critically reflect on new knowledge or understanding gained, how their thinking, behaviour or opinions have changed or developed since the beginning of the process, and the importance of this.

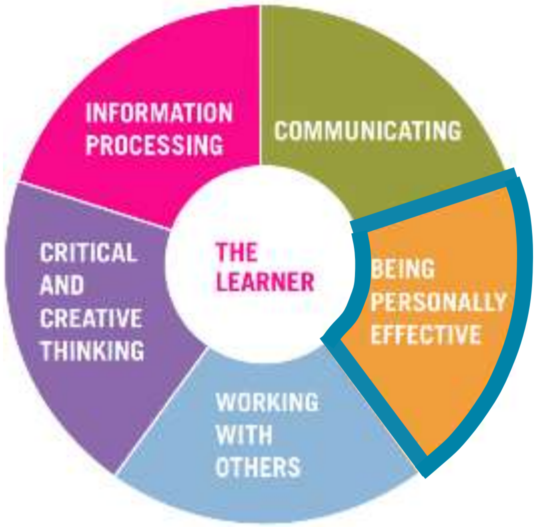

Coimisiún na Scrúduithe Stáit
State Examinations Commission

Leaving Certificate Coursework

Information note for four Leaving Certificate subjects with new subject specifications

- Agricultural Science – Individual Investigative Study
- Computer Science – Coursework Project
- Economics – Student Research Project
- Physical Education – Physical Activity Project

The Modelling Cycle
Creating a Timeline



In groups, discuss an appropriate timeline for students’ engagement with the project and how teachers will support them during this timeframe.

How can you draw on the mathematical modelling methods used in your classroom to support students’ engagement with the project?

Reflection

What were your key takeaways from this session?

How can you implement ideas from this session into your teaching?

What are the next steps for enhancing students’ modelling skills in your classroom?

Tea/Coffee Break

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Dynamic Programming with Multi-Stage Authentic Problems

11:15 – 13:00

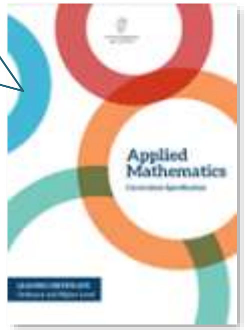
By The End of This Session You Will Have:

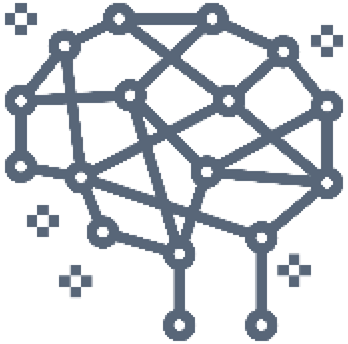
Explored the use of a Concepts through Modelling approach to developing understanding of Dynamic Programming as applied to multi-stage authentic problems.

An understanding of differences between algorithms and the appropriate use of each in terms of their correctness and their ability to yield an optimal solution.

Resources
Strand 2 Support

“Use algorithms to solve problems.”
p. 17, specification.





Seminar 1: **Introduction to Networks and Graph Theory, Algorithms and their Applications**

Seminar 2: **Development of Dijkstra’s Algorithm through Modelling**

Seminar 4: **Project Scheduling**

Seminar 5: **Bellman’s Principle of Optimality and Dynamic Programming**

Seminar 8: **Exploring Project Scheduling with Project Scheduling Diagrams**

All slides and relevant resources available on the Applied Mathematics section of the website under CPD Resources.

https://pdst.ie/post-primary/sc/appliedmaths/cpd-resources

Strand 2 Algorithms

Minimum Spanning Tree

Prim's

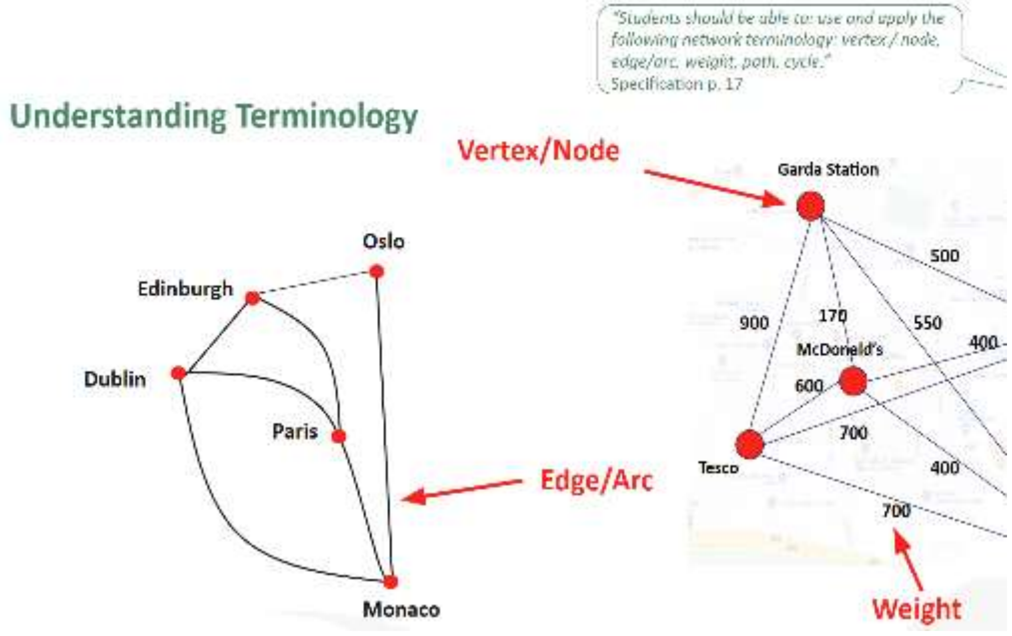
Starts from a single vertex and adds edges one at a time

Generally faster for dense graphs

Kruskal's

Sorts edges by weight and adds them to the tree if they don't create a cycle

Works well with sparse graphs, does not require a starting vertex



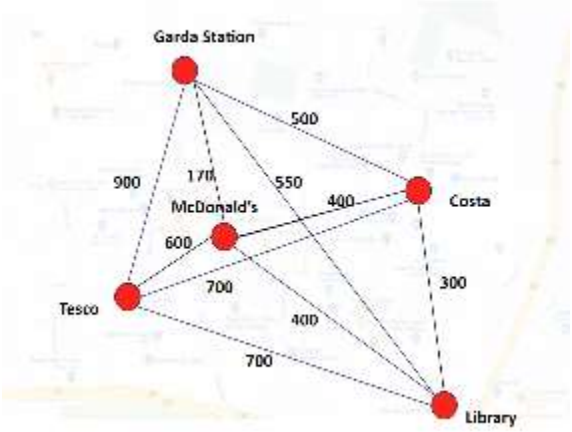
Oide

Strand 2 Algorithms
Minimum Spanning Tree

Recall: NS 1

Concepts through Modelling Approach

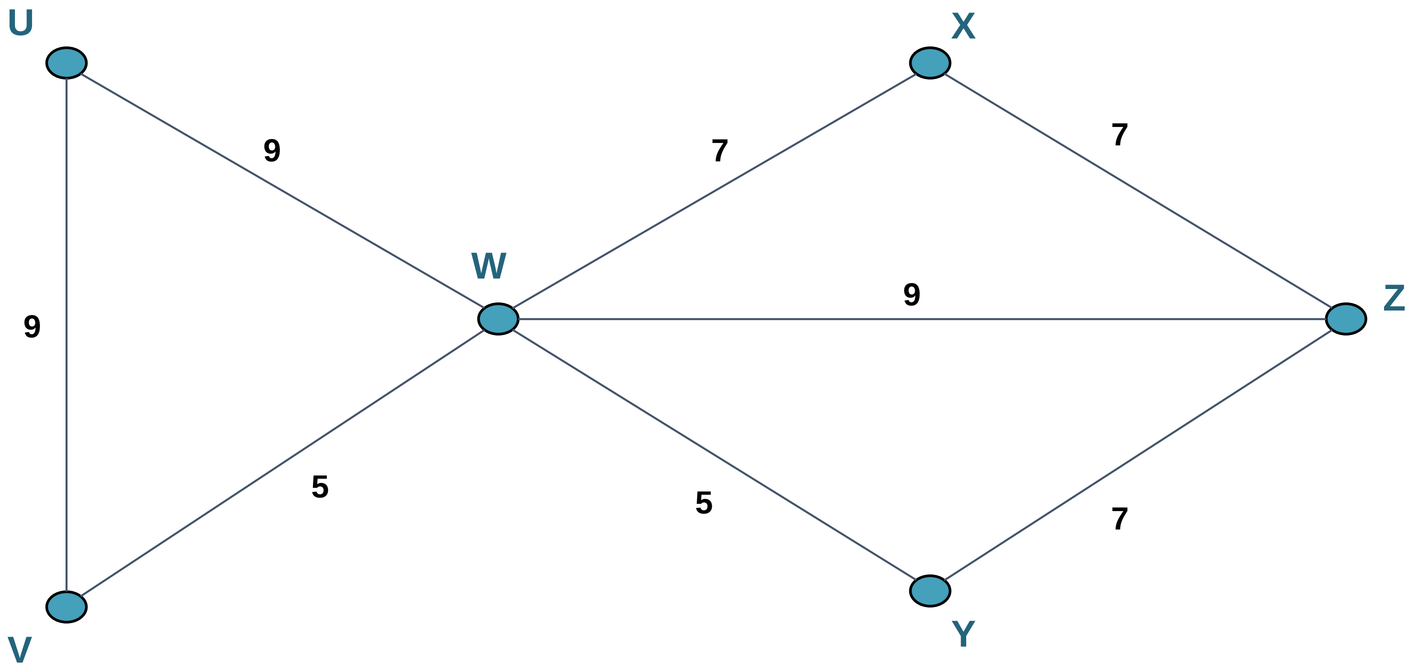
Broadband for Mallow. Buildings are connected by laying cables in the ground following the current road layout. We used **Prim's** and **Kruskal's** to investigate.



Strand 2 Algorithms

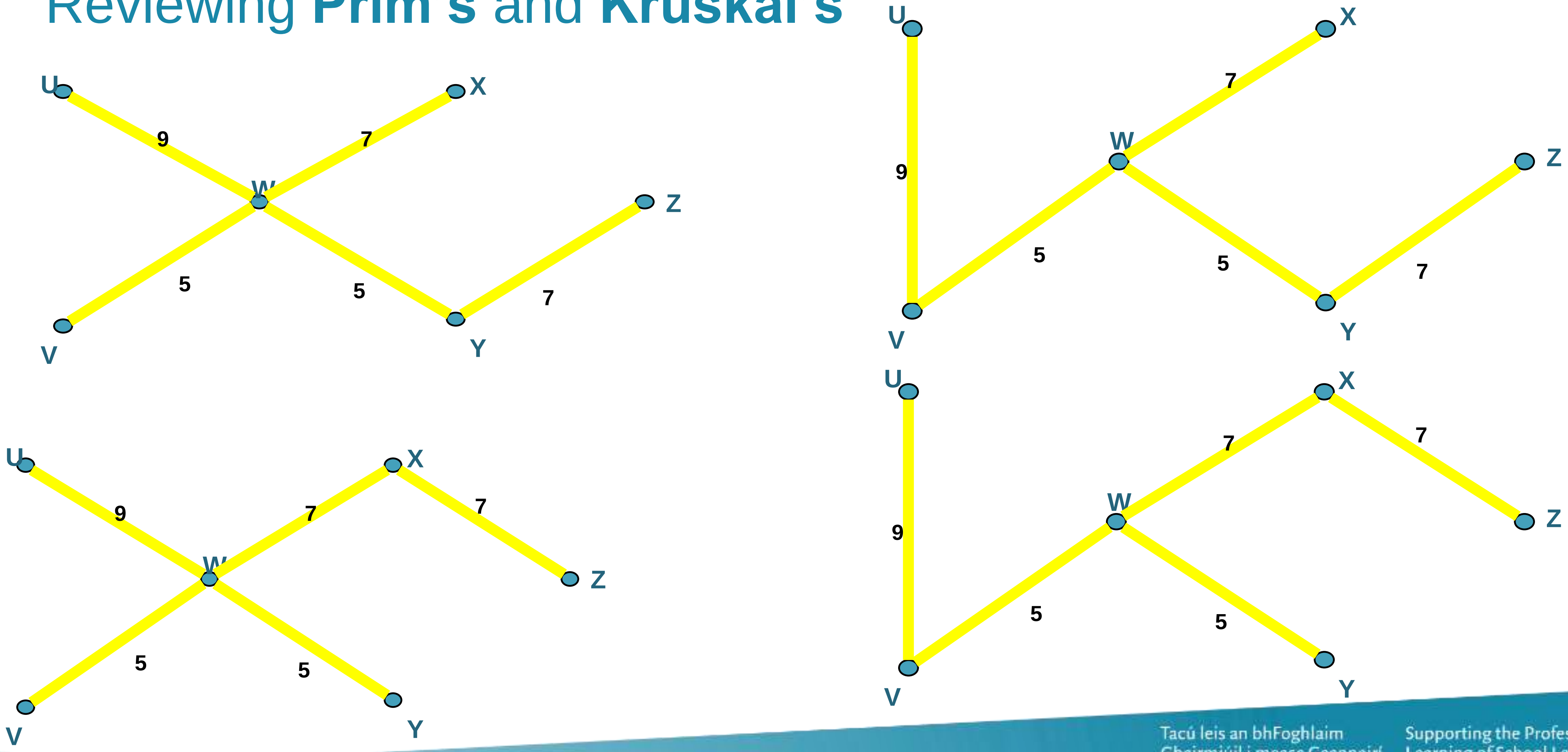
Reviewing **Prim's** and **Kruskal's**

Find a minimum spanning tree for the below network using Prim's and then Kruskal's Algorithm. There are many possible solutions.



Strand 2 Algorithms

Reviewing **Prim's** and **Kruskal's**



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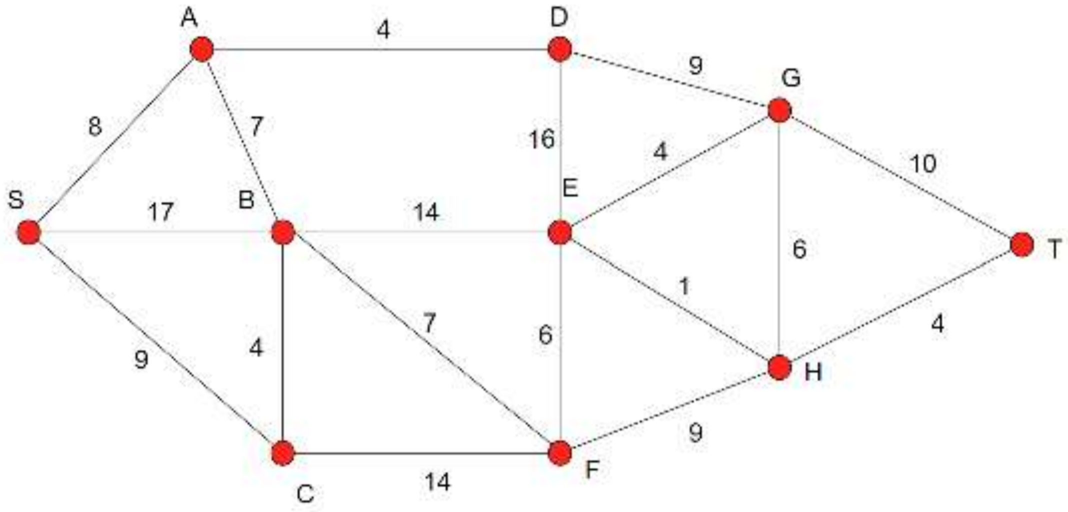
Strand 2 Algorithms

Recall: NS 2

Concepts through Modelling Approach

Road network with each weight representing distance in km. Use **Dijkstra's** Algorithm to find the shortest route.



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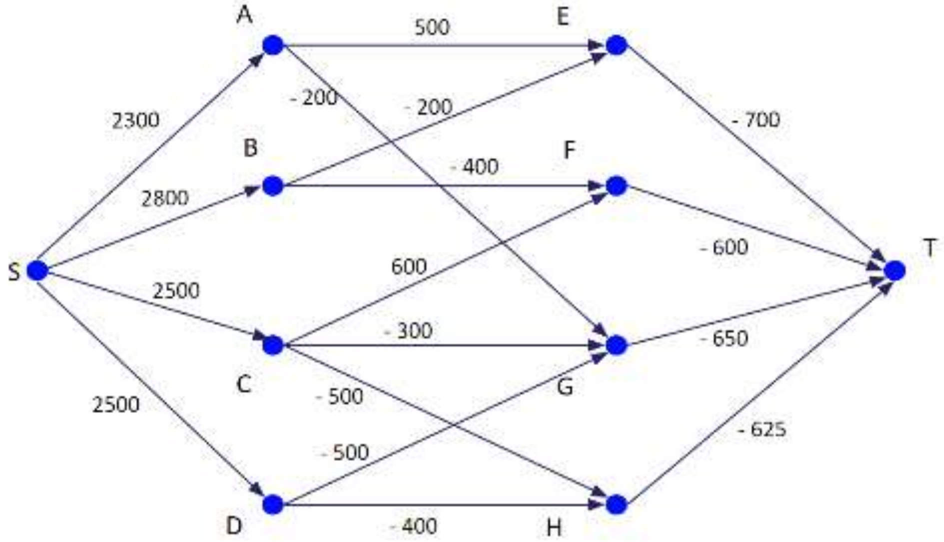
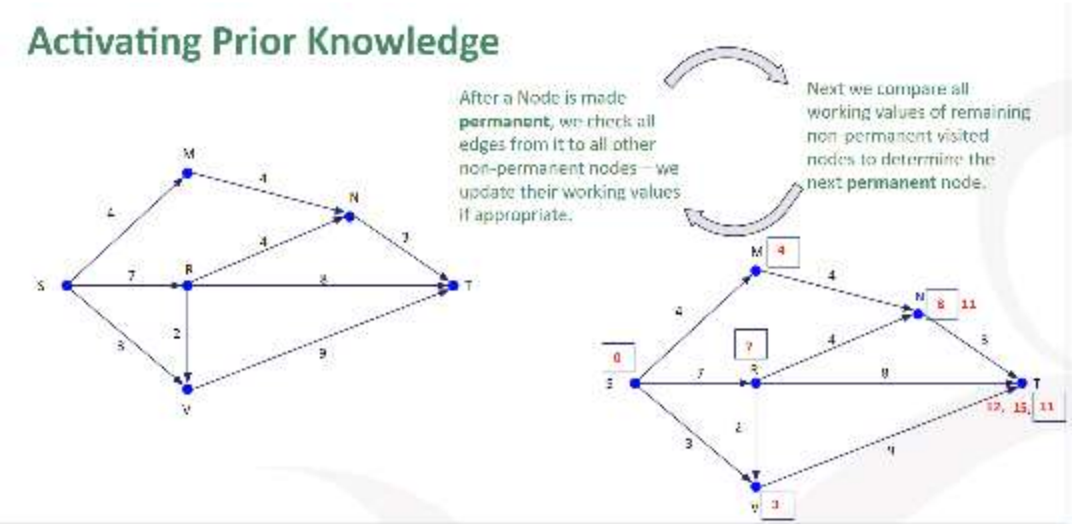
Recall: NS 5

Concepts through Modelling Approach

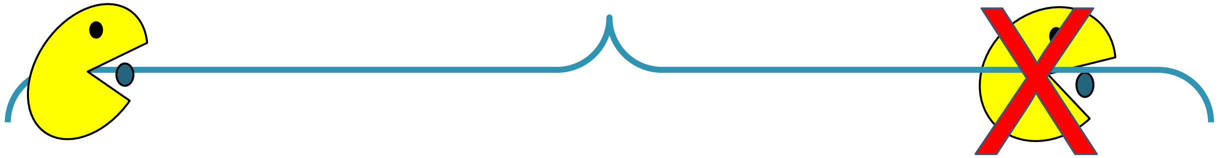
Transition Year school tour to Austria, we used **Bellman's** to investigate the best package based on cost and activities for each student.



Strand 2 Algorithms



Optimization



Dijkstra's

Finds the shortest path between a source vertex and all other vertices

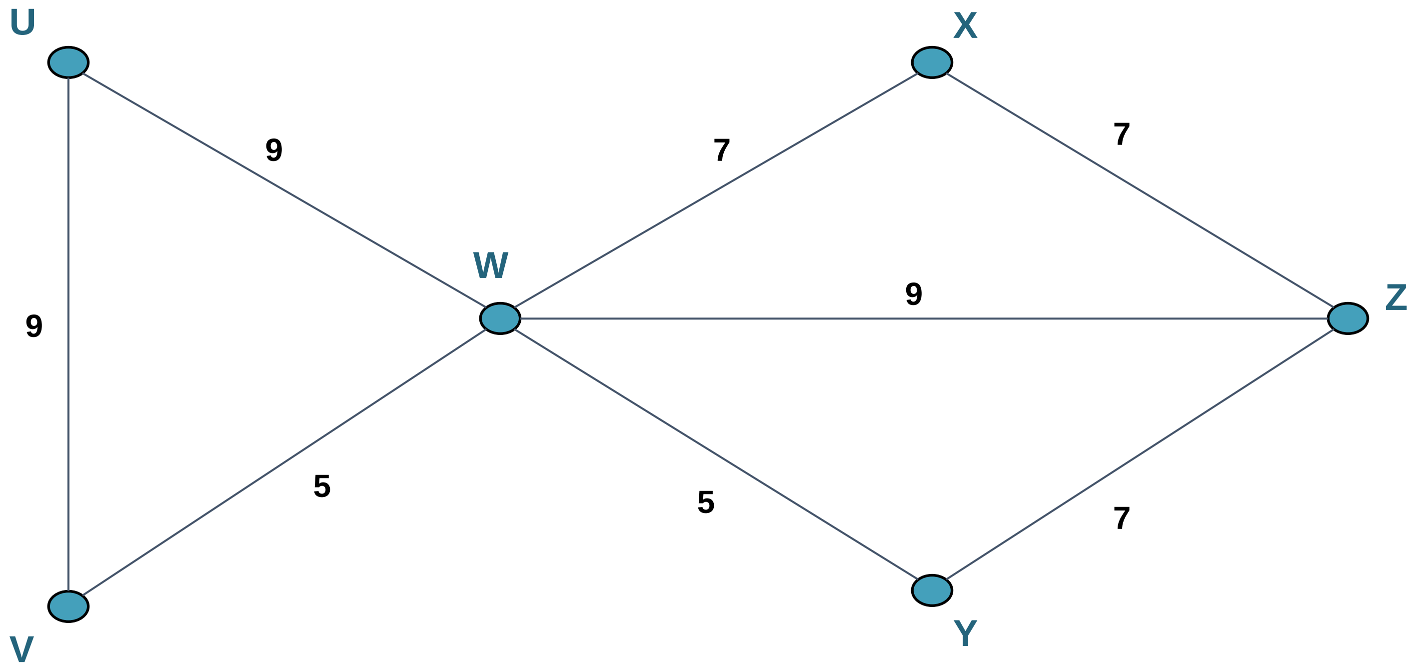
Breaks down with negative edge weights

Dynamic Programming

Breaks a problem down into smaller sub-problems. Solutions to sub-problems stored and then the solution to overall problem constructed from the solutions to the sub-problems.

Strand 2 Algorithms
Reviewing **Dijkstra's** Algorithm

Apply Dijkstra's algorithm to find the shortest path from U to Z.



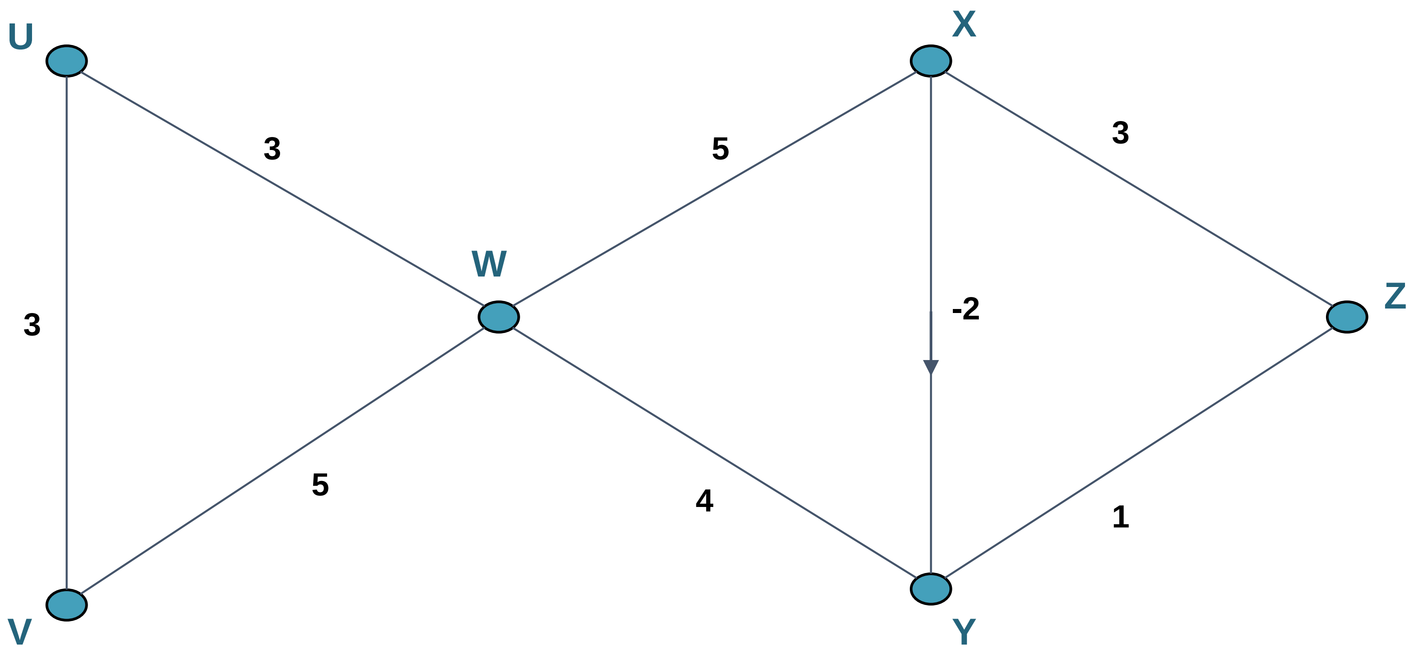
Strand 2 Algorithms

Reviewing **Dijkstra's** Algorithm

UWZ with the lowest possible weight of 18.

Strand 2 Algorithms
Reviewing **Dijkstra's** Algorithm

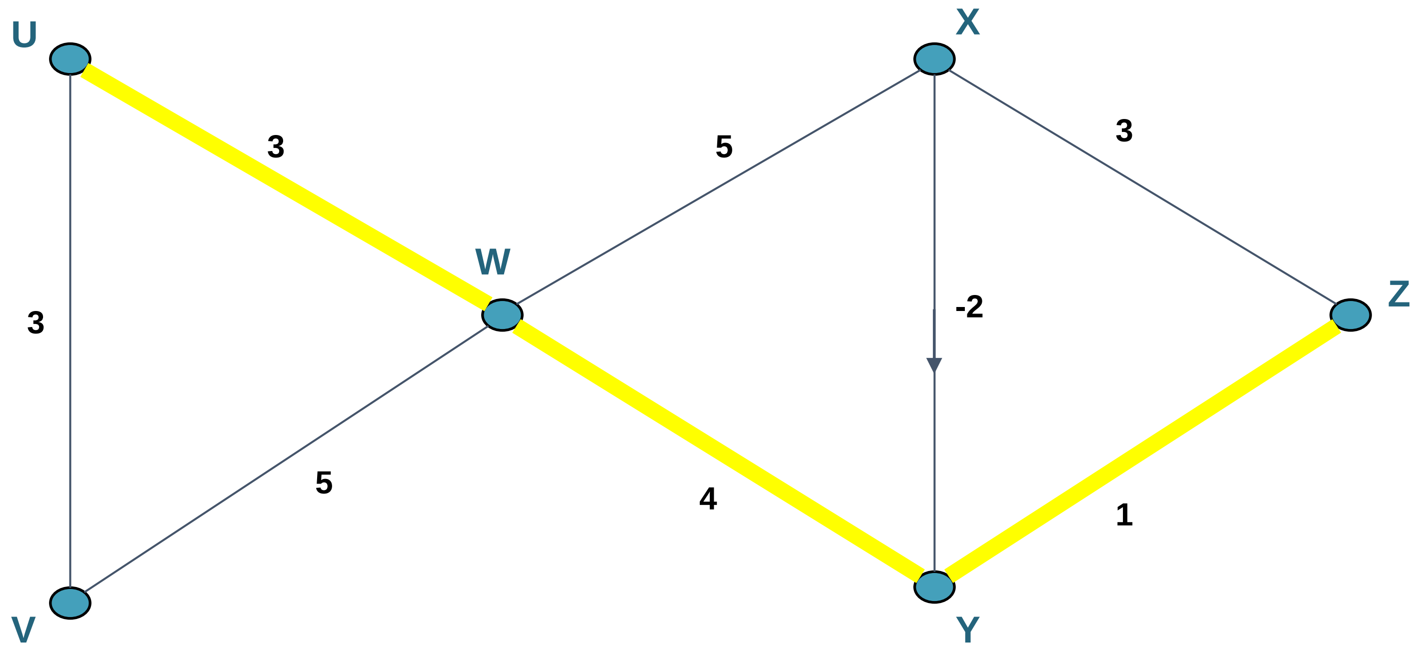
Apply Dijkstra's to find the shortest path from U to Z in this network. Does it yield a correct solution?



Strand 2 Algorithms

Dijkstra's Algorithm Shortest Path

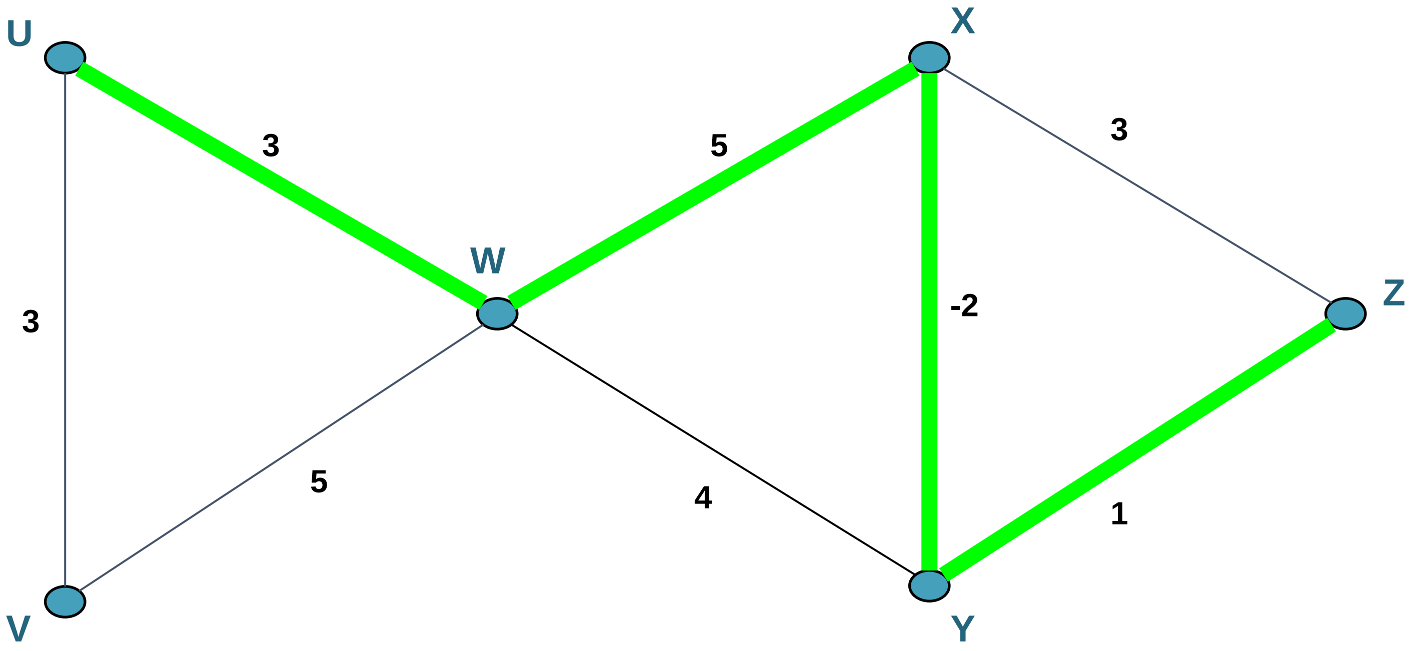
Applying Dijkstra's gives a shortest path of UWYZ with a total weight of 8. Is this correct, Is there a shorter path?



Strand 2 Algorithms

Dijkstra's Algorithm Shortest Path

The total weight of path UWXYZ is 7 (3 + 5 -2 +1). This is the shortest path, In this instance Dijkstra yielded a sub-optimal solution. Other non-greedy more versatile algorithms may be required depending on the type of problem.



Strand 2 Algorithms
Dynamic Programming

"Dynamic Programming and shortest paths as applied to multi-stage authentic problems."
p. 17, specification.

- Dynamic Programming is not greedy
- Uses backward recursion it takes an overall view of a probler
- Can handle maximum and minimum problems easily and negative edge weights.
- Easily applicable to problems given in the form of a table.

Main disadvantages: requires a staged network and as it stores sub-problems, the time cost and space required to implement are higher.

Strand 2 Algorithms

Applying **Dynamic Programming**

Dynamic Programming is based on Bellman’s Principle of Optimality

Any part of the shortest/longest path between the source and sink nodes is itself a shortest/longest path

Or: ‘any part of the optimal path is itself optimal’

Approaches To Mathematical Modelling in The Classroom

Recall

1

Concepts then Modelling

Explore a number of mathematical concepts through suitable tasks, word problems etc., then solve a rich modelling problem. In exploring these tasks, modelling competences may also be developed.

Complete a full modelling cycle.

Focus on a subset of competences.

2

Concepts through Modelling

Explore a rich modelling problem and, as the need arises, develop understanding of new mathematical concepts through instruction, guided discovery, research, etc.

Complete a full modelling cycle.

Focus on a subset of competences.

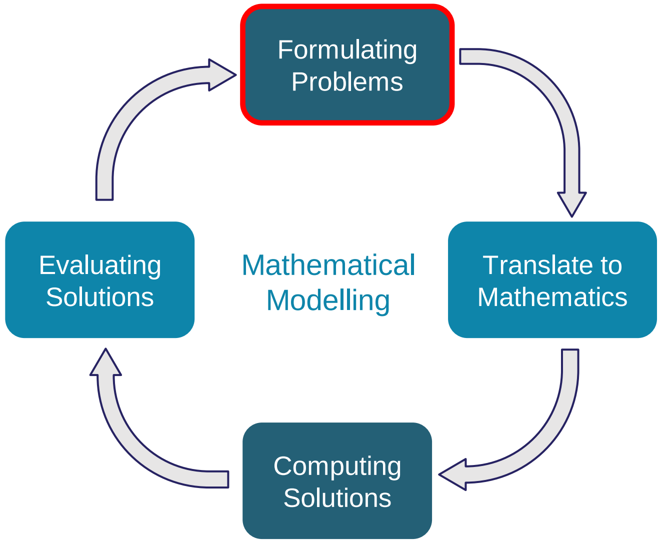
Interpreting a Real-World Problem

2

Concepts through Modelling

In many real-world settings the management of stock is an important consideration.

Choose a real-world problem related to the distribution or management of stock and model the problem(s) you have selected using *The Modelling Cycle*.



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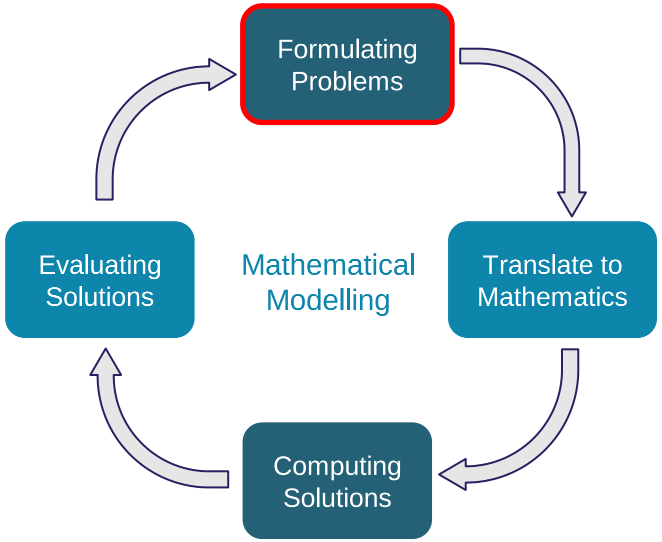
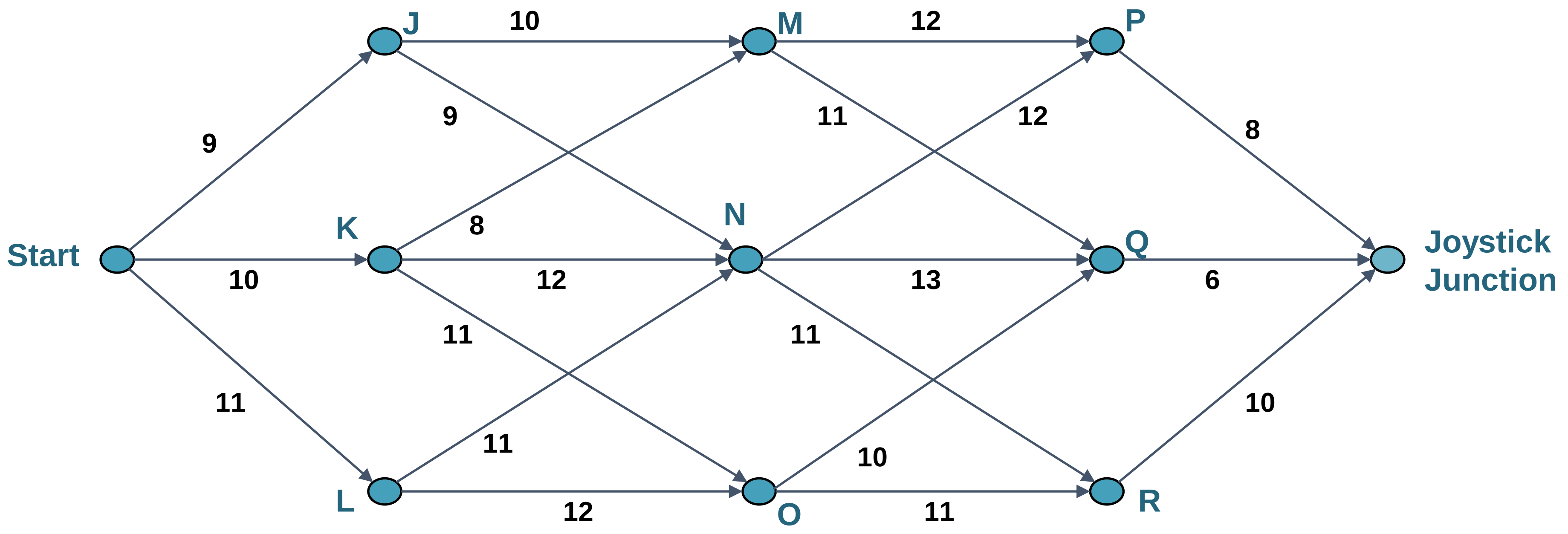
Interpreting a Real-World Problem
Formulating Problems

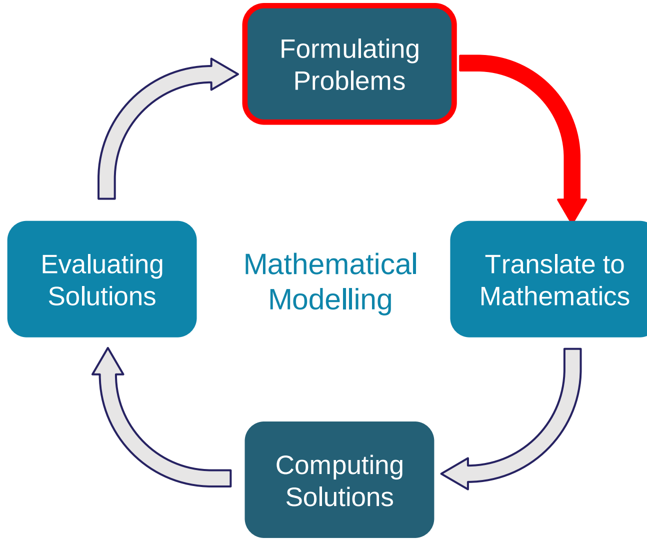
Problem Statement: *Joystick Junction* has the last remaining stock of a new games console. What is the best route to take to get to Joystick Junction on the other side of the city?



The Modelling Cycle
Translating to Mathematics

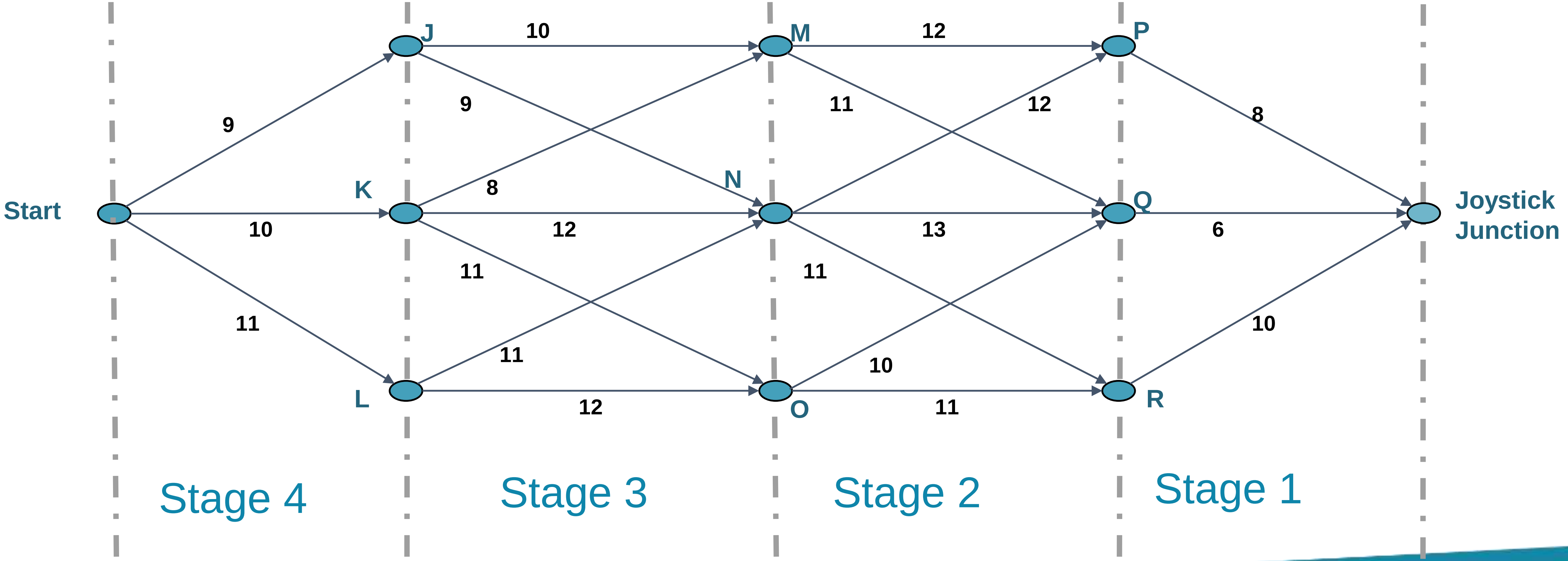
The city can be represented with a simplified network. The values on each edge represent journey time in minutes from each vertex or node to the next and each of the nodes.





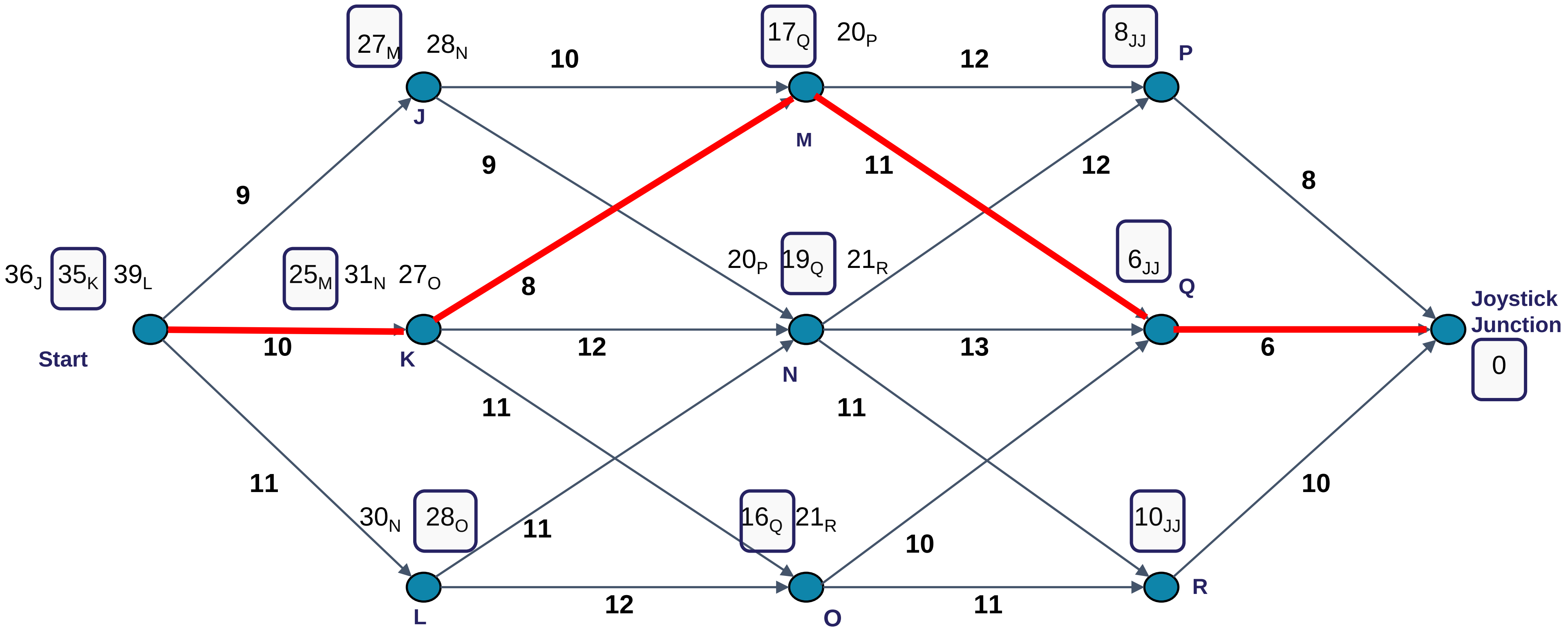
Computing Solutions

Identify the stages



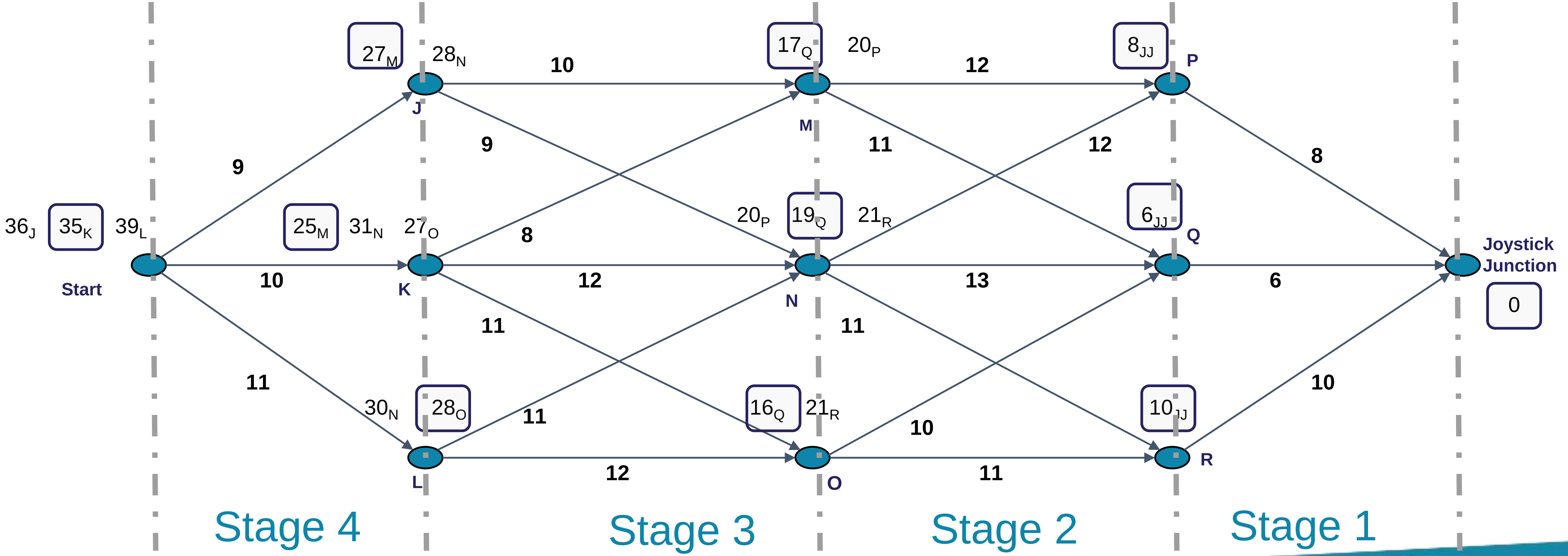
Computing Solutions

Applying Bellman's Principle directly to the network



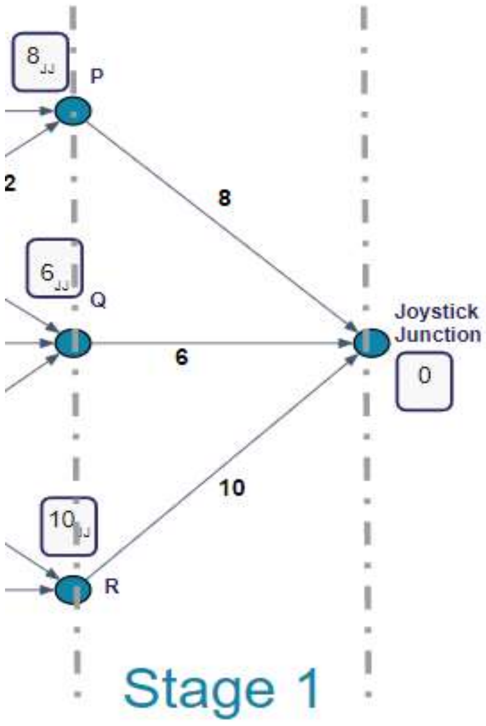
Computing Solutions

Applying Bellman's Principle using a Table



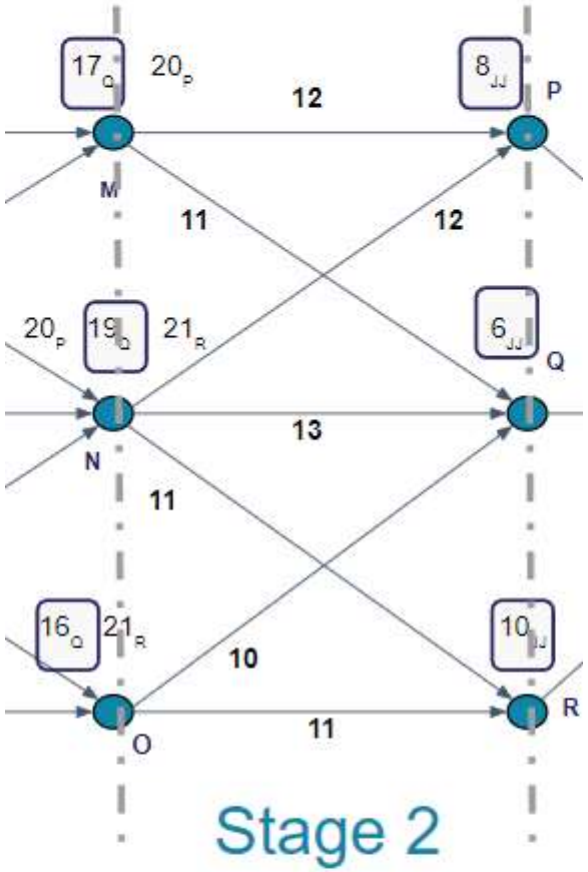
Computing Solutions

Applying Bellman's Principle using a Table



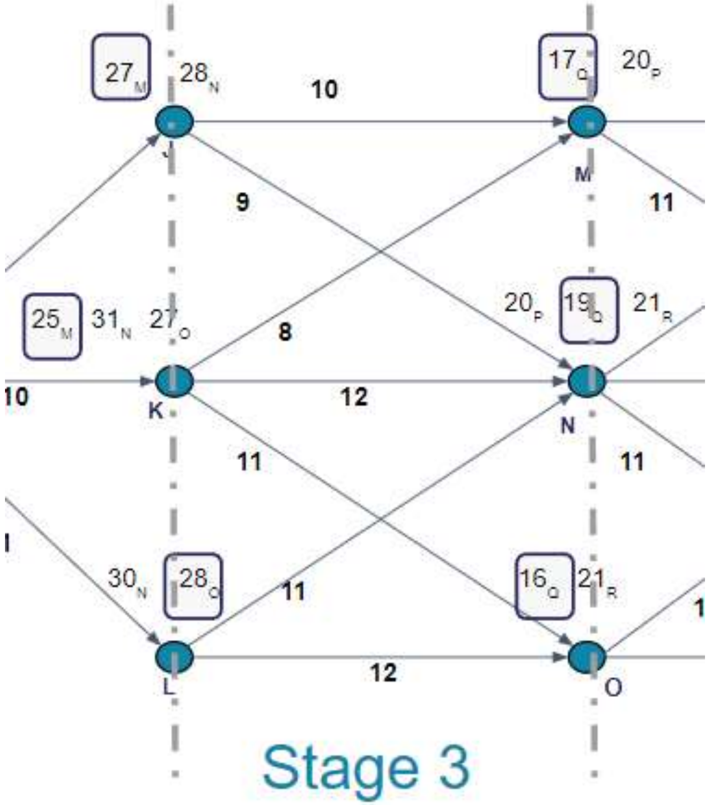
Stage	State (Vertex)	Action	Value
1	P	P-Joystick Junction	8*
	Q	Q-Joystick Junction	6*
	R	R-Joystick Junction	10*

Computing Solutions



Stage	State (Vertex)	Action	Value
2	M	MP	12 + 8 = 20
		MQ	11 + 6 = 17*
	N	NP	12 + 8 = 20
		NQ	13 + 6 = 19*
		NR	11 +10 = 21
	O	OQ	10 + 6 = 16*
		OR	11 +10 = 21

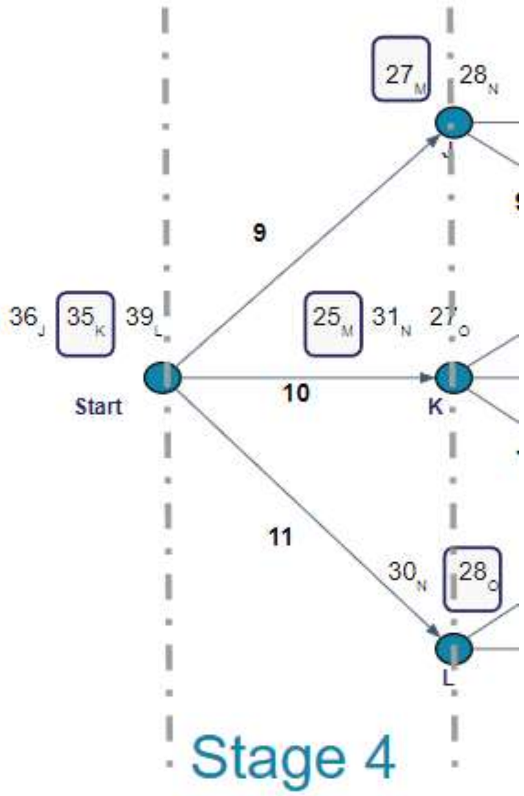
Computing Solutions



Stage	State (Vertex)	Action	Value
3	J	JM	10 + 17 = 27*
		JN	9 + 19 = 28
	K	KM	8 + 17 = 25*
		KN	12 + 19 = 31
		KO	11 +16 = 27
	L	LN	11 + 19 = 30
		LO	12 +16 = 28*

Computing Solutions

What is the shortest route to Joystick Junction?



Stage	State (Vertex)	Action	Value
4	Start	Start-J	9 + 27 = 36
		Start-K	10 + 25 = 35*
		Start-L	11 + 28 = 39

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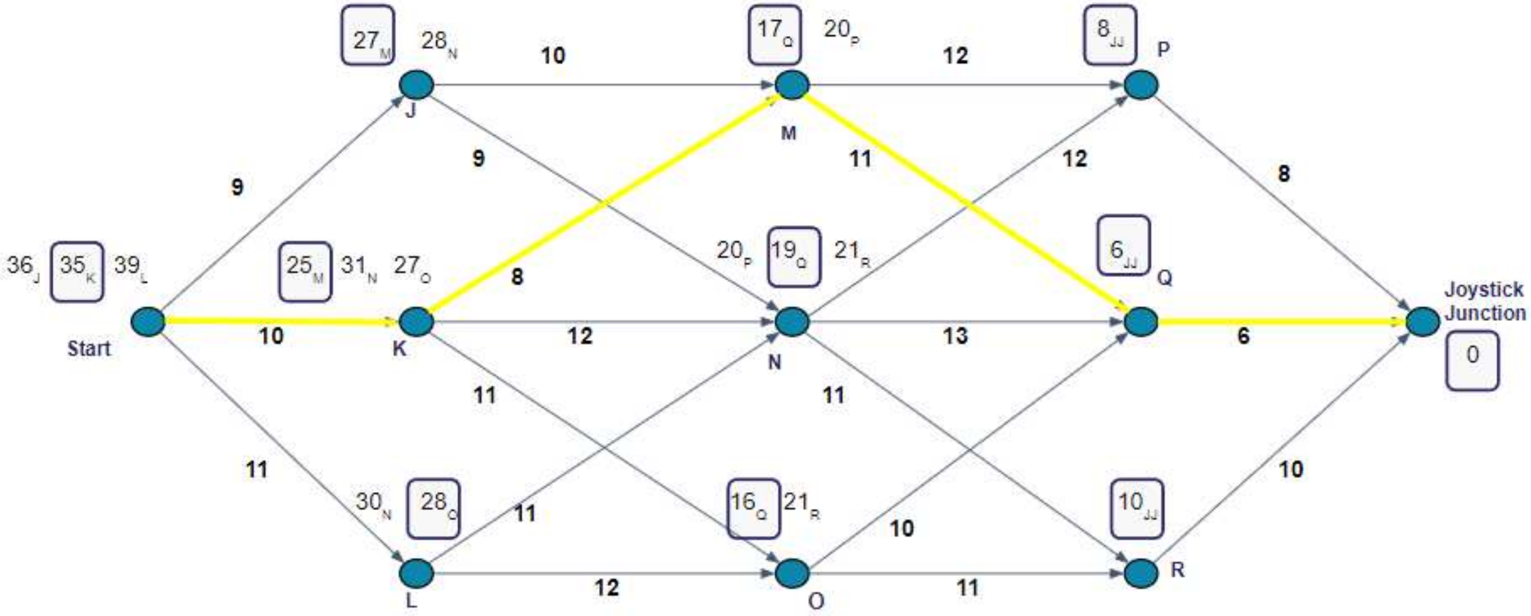
Stage	State	Action	Value
1	P	JJ	0+8 = 8*
	Q	JJ	0+6 = 6*
	R	JJ	0+10 = 10*
2	M	P	12 + 8 = 20
		Q	11+6 = 17*
	N	P	12+8 = 20
		Q	13+6 = 19*
		R	10+11 = 21
	O	Q	10+6 = 16*
		R	11+10 = 21
3	J	M	10+17 = 27*
		N	9+19 = 28
	K	M	8+17 = 25*
		N	12+19 = 31
		O	11+16 = 27
	L	N	11+19 = 30
		O	12+16 = 28*
4	Start	J	9+27 = 36
		K	10+25 = 35*
		L	11+28 = 39



The shortest route is :

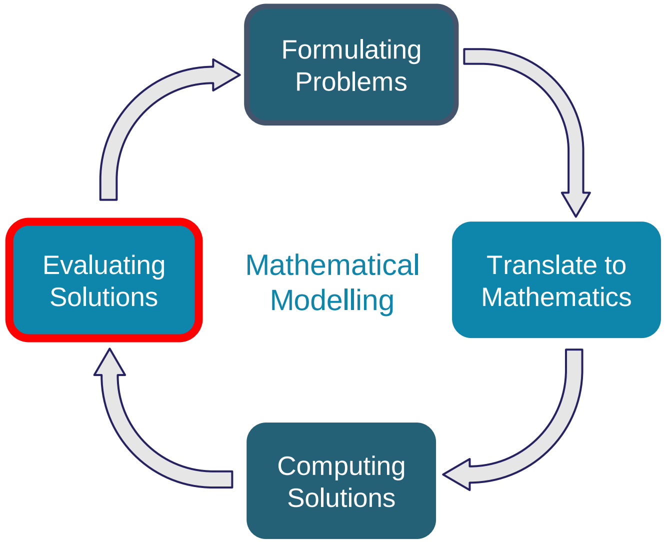
Start - K - M - Q - Joystick Junction

Evaluating Solutions

Interpret your mathematical solution(s) in the context of the problem you are modelling.

How accurate and reliable is your solution based on your earlier assumptions?

How can you refine your assumptions to improve your solution and how will this change your solution?



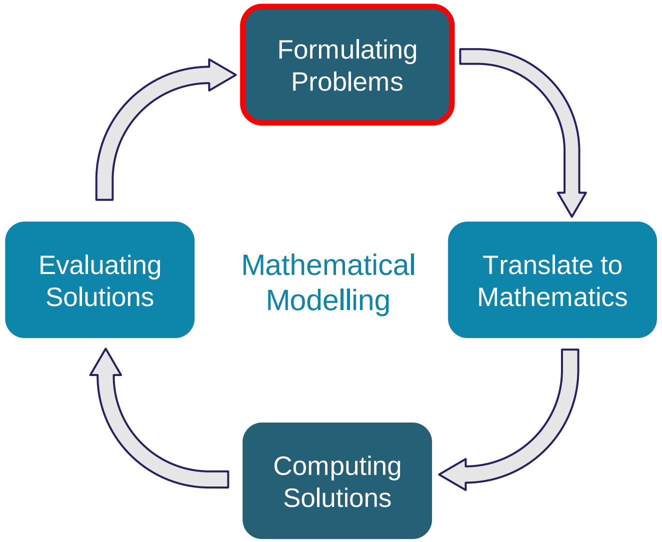
Interpreting a Real-World Problem
Formulating Problems

2

Concepts through Modelling

In many real-world settings the management of stock is an important consideration.

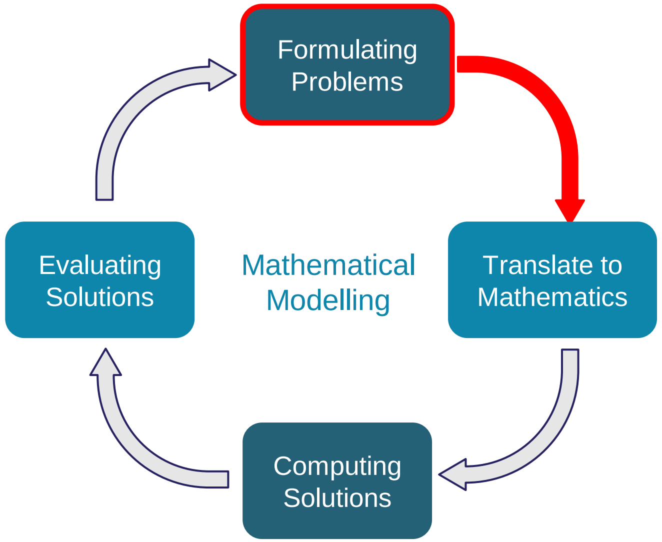
Choose a real-world problem related to the distribution or management of stock and model the problem(s) you have selected using *The Modelling Cycle*.



The Modelling Cycle
Formulating Problems

A games manufacturer needs to distribute 500 games consoles every month and can allocate these in multiples of 100 to three different retailers. The distributor fee/profit, in €100s, for the number of units allocated to each retailer is shown in the table.

Number of consoles allocated	**100**	**200**	**300**	**400**	**500**
Joystick Junction	€11	€25	€30	€32	€33
Button Bashers	€15	€18	€19	€20	€21
Gamers Grotto	€7	€14	€21	€28	€35



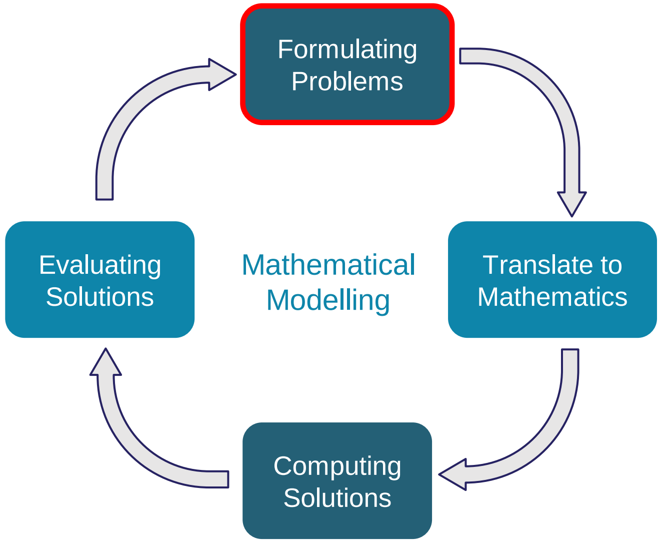
The manufacturer wants to know how many consoles should be allocated to each retailer to maximise their monthly income.

Interpreting a Real-World Problem
Formulating Problems

Problem Statement: How should I allocate stock across a number of retailers to maximise profit?

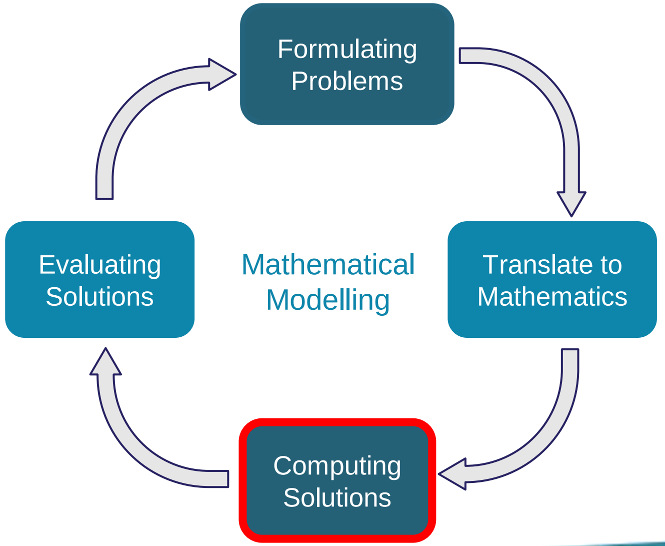
2

Concepts through Modelling



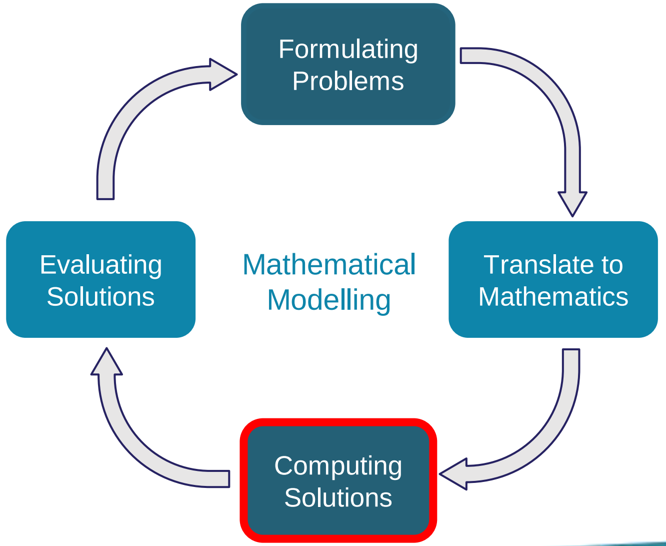
Stage	State (units available x100)	Action (units allocated x100)	Destination (units remaining x100)	Value (cumulative profit x100)
1 Gamers Grotto	0	0	0	0*
	1	1	0	7*
	2	2	0	14*
	3	3	0	21*
	4	4	0	28*
	5	5	0	35*

Number of consoles allocated	100	200	300	400	500
Joystick Junction	€11	€25	€30	€32	€33
Button Bashers	€15	€18	€19	€20	€21
Gamers Grotto	€7	€14	€21	€28	€35



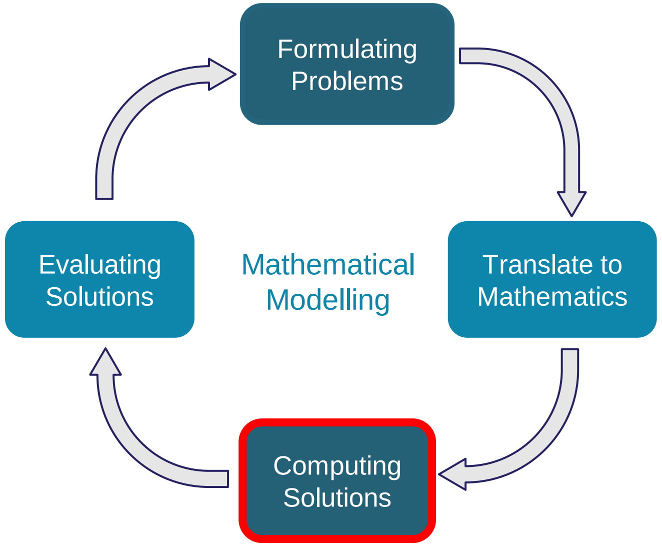
Stage	State (units available)	Action (units allocated)	Destination (units remaining)	Value (cumulative profit)
2 Button Bashers	0	0	0	0+ 0 = 0*
	1	1	0	15 + 0 = 15*
		0	1	0 + 7 = 7
	2	2	0	18 + 0 = 18
		1	1	15 + 7 = 22*
		0	2	0 + 14 = 14

Number of consoles allocated	100	200	300	400	500
Joystick Junction	€11	€25	€30	€32	€33
Button Bashers	€15	€18	€19	€20	€21
Gamers Grotto	€7	€14	€21	€28	€35



Stage	State (units available x100)	Action (units allocated x100)	Destination (units remaining x100)	Value (cumulative profit x100)
2 Button Bashers	3	3	0	19 + 0 = 19
		2	1	18 + 7 = 25
		1	2	15 + 14 = 29*
		0	3	0 + 21 = 21
	4	4	0	20 + 0 = 20
		3	1	19 + 7 = 26
		2	2	18 + 14 = 32
		1	3	15 + 21 = 36*
		0	4	0 + 28 = 28

Number of consoles allocated	100	200	300	400	500
Joystick Junction	€11	€25	€30	€32	€33
Button Bashers	€15	€18	€19	€20	€21
Gamers Grotto	€7	€14	€21	€28	€35



Stage	State (units available x100)	Action (units allocated x100)	Destination (units remaining x100)	Value (cumulative profit x100)
2 Button Bashers	5	5	0	21 + 0 = 21
		4	1	20 + 7 = 27
		3	2	19 + 14 = 33
		2	3	18 + 21 = 39
		1	4	15 + 28 = 43*
		0	5	0 + 35 = 35

Number of consoles allocated	100	200	300	400	500
Joystick Junction	€11	€25	€30	€32	€33
Button Bashers	€15	€18	€19	€20	€21
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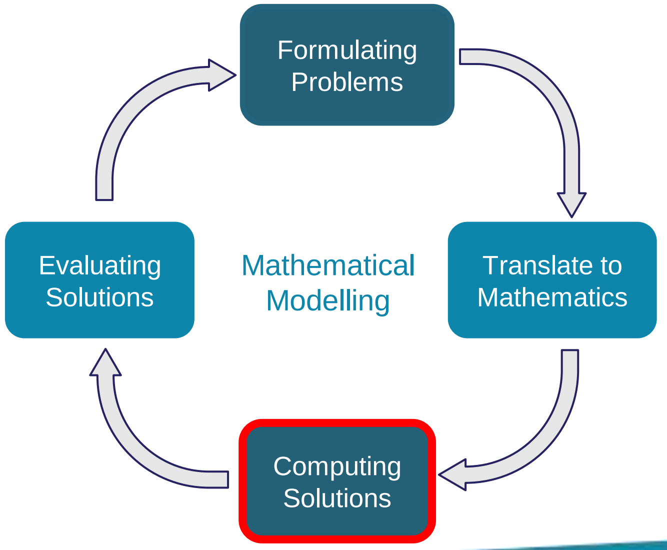
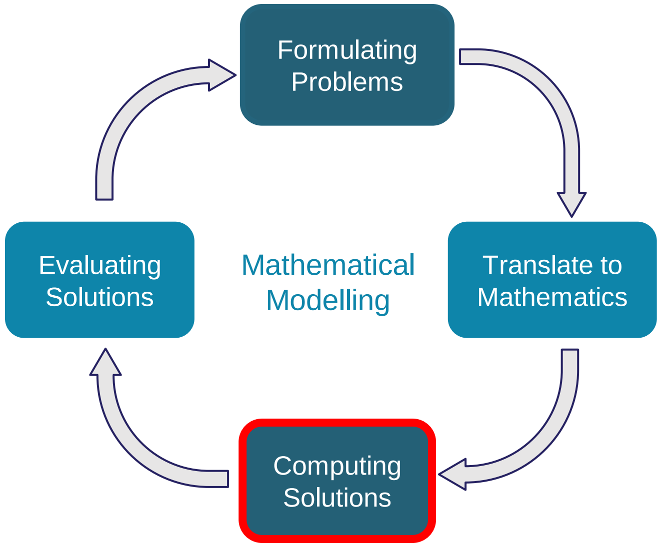


Stage	State (units available)	Action (units allocated)	Destination (units remaining)	Value (cumulative profit)
3 Joystick Junction	5	5	0	33 + 0 = 33
		4	1	32 + 15 = 47
		3	2	30 + 22 = 52
		2	3	25 + 29 = 54*
		1	4	11 + 36 = 47
		0	5	0 + 43 = 43

Number of consoles allocated	100	200	300	400	500
Joystick Junction	€11	€25	€30	€32	€33
Button Bashers	€15	€18	€19	€20	€21
Gamers Grotto	€7	€14	€21	€28	€35



Stage	State (Units Available)	Action (Units Allocated)	Destination (Units Remaining)	Value (Cumulative Profit)
Gamers Grotto	0	0	0	0*
	1	1	0	7*
	2	2	0	14*
	3	3	0	21*
	4	4	0	28*
	5	5	0	35*
Button Bashers	0	0	0	0 + 0 = 0*
	1	1	0	15 + 0 = 15*
		0	1	0 + 7 = 7
	2	2	0	18 + 0 = 18
		1	1	15 + 7 = 22*
		0	2	0 + 14 = 14
	3	3	0	19 + 0 = 19
		2	1	18 + 7 = 25
		1	2	15 + 14 = 29*
		0	3	0 + 21 = 21
	4	4	0	20 + 0 = 20
		3	1	19 + 7 = 26
		2	2	18 + 14 = 32
		1	3	15 + 21 = 36*
		0	4	0 + 28 = 28
	5	5	0	21 + 0 = 21
		4	1	20 + 7 = 27
		3	2	19 + 14 = 33
		2	3	18 + 21 = 39
		1	4	15 + 28 = 43*
		0	5	0 + 35 = 35
Joystick Junction	5	5	0	33 + 0 = 33
		4	1	32 + 15 = 47
		3	2	30 + 22 = 52
		2	3	25 + 29 = 54*
		1	4	11 + 36 = 47
		0	5	0 + 43 = 43

Oide

To maximise the distributor fees/profit, the best way to allocate the 500 consoles is to distribute

200 consoles to Joystick Junction, 100 consoles to Button Bashers and 200 consoles to Gamers Grotto.

Number of consoles allocated	100	200	300	400	500
Joystick Junction	€11	€25	€30	€32	€33
Button Bashers	€15	€18	€19	€20	€21
Gamers Grotto	€7	€14	€21	€28	€35

Reflection

What were your key takeaways from this session?

What considerations are needed to take this learning back to your classroom?

Lunchtime

Oide

Tacú leis an bhFoghlaim Ghairmiúil i measc Ceannairí Scoile agus Múinteoirí

Supporting the Professional Learning of School Leaders and Teachers

Exploring Difference Equations

14:00 – 15:30

By The End Of This Session You Will Have:

Engaged with and discussed suited range of pedagogical approaches to developing an understanding of when to use difference equations.

Explored using prior knowledge to gain an understanding of how difference equations may be developed and then formalised through authentic modelling problems.

Approaches to Mathematical Modelling in the Classroom

Recall

1

Concepts, then Modelling

Explore a number of mathematical concepts through suitable tasks, word problems etc., then solve a rich modelling problem. In exploring these tasks, modelling competences may also be developed.

2

Concepts through Modelling

Explore a rich modelling problem and, as the need arises, develop understanding of new mathematical concepts through instruction, guided discovery, research, etc.

Prior Knowledge
Difference Equations

A Recurrence relation is an equation that defines a sequence where the next term is a function of the previous term(s).

4, 7, 12, 19, 28, 39, ….

0, 1, 3, 14, 57, 227, 966, ….

0, 1, 1, 2, 3, 5, 8, 13, 21 ….

1, 2, 2, 4, 8, 32, 256, ….

This mathematical relationship often involves the **differences between successive values** of a function of a discrete variable – hence the expression *Difference equations.*

Prior Knowledge

Recall - Word Problem from National Seminar 3

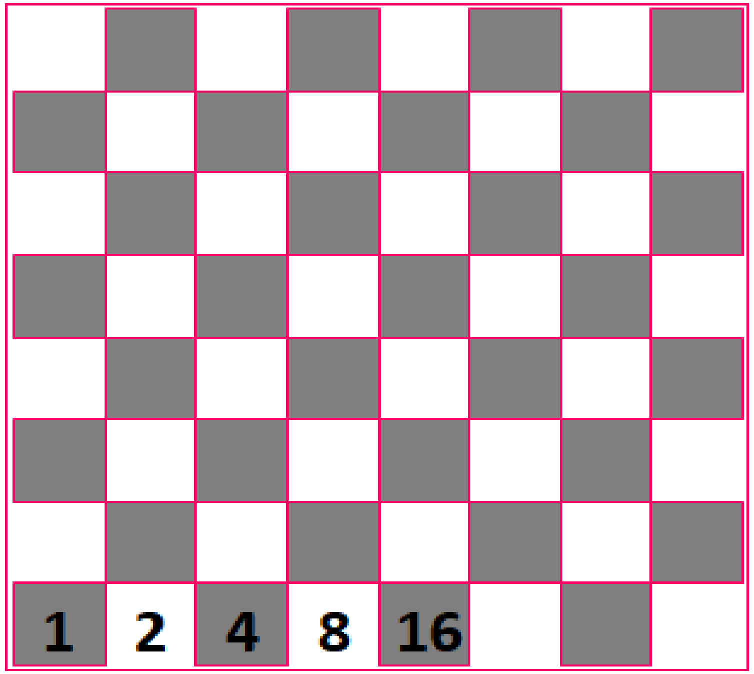
According to legend King Shirham of India wanted to reward his servant for inventing and presenting him with the game of chess. The desire of his servant seemed modest: “Give me a grain of wheat to put on the first square of this chessboard, and two grains to put on the second square, and four grains to put on the third, and eight grains to put on the fourth and so on, doubling for each successive square, give me enough grain to cover all 64 squares.”

“You don’t ask for much. Your wish will certainly be granted” exclaimed the king.

Based on an extract from “One, Two, Three…Infinity”, Dover Publications

Prior Knowledge

Junior Certificate Mathematics



$T_1 = 1$ $\quad$ $T_1 = 1 = 2^0$

$T_2 = 2$ $\quad$ $T_2 = 2 = 2^1$

$T_3 = 4$ $\quad$ $T_3 = 4 = 2^2$

$T_4 = 8$ $\quad$ $T_4 = 8 = 2^3$

$T_5 = 16$ $\quad$ $T_5 = 16 = 2^4$

$$S_{64} = 2^0 + 2^1 + 2^2 + 2^3 + 2^4 + \dots + 2^{63}$$

$$\mathbf{2 \times S_{64}} = 2^1 + 2^2 + 2^3 + 2^4 + \dots + 2^{63} + 2^{64}$$

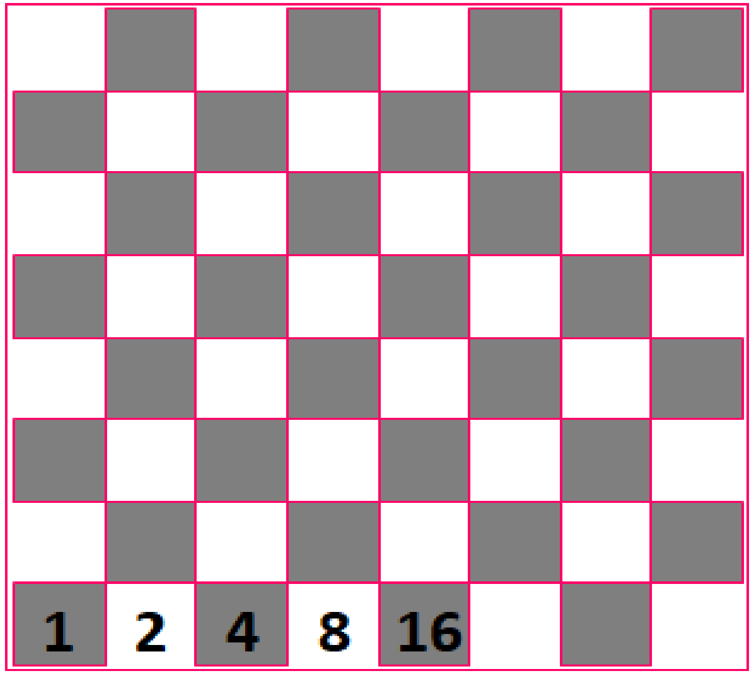
$$\mathbf{-S_{64}} = -(2^0 + 2^1 + 2^2 + 2^3 + 2^4 + \dots + 2^{63})$$

$$= 2^{64} - 2^0 = 2^{64} - 1$$

$$S_{64} = 2^{64} - 1$$

Prior Knowledge

Leaving Certificate Mathematics



$T_1 = 1$ $\quad$ $T_1 = 1 = 2^0$

$T_2 = 2$ $\quad$ $T_2 = 2 = 2^1$

$T_3 = 4$ $\quad$ $T_3 = 4 = 2^2$

$T_4 = 8$ $\quad$ $T_4 = 8 = 2^3$

$T_5 = 16$ $\quad$ $T_5 = 16 = 2^4$

Recurrence relation $T_n = 2^{n-1}$ $n \varepsilon N, n>1$

$S_{64} = 1 + 2^1 + 2^2 + 2^3 + 2^4 + \dots + 2^{63}$

This a geometric series
The first term a =1, the ratio r = 2

$$S_n = \frac{a(r^n - 1)}{(r-1)} \Rightarrow S_{64} = \frac{1(2^{64} - 1)}{(2-1)}$$

$$S_{64} = (2^{64} - 1)$$

What Decides the Order of an Equation?

Consider the sequence of numbers: 0, 1, 1, 2, 3, 5, 8, 13, 21 ….

Recurrence relation $U_{n+1} = U_n + U_{n-1}$

$n>1,\ n \varepsilon N$

Order = *difference between the iterates*

= *(n+1)-(n-1)*

=> *Order* **= 2**

This equation is called **homogeneous** because each term is determined by its previous terms only.

Determine the order of the following

Difference Equation	Order of the Equation	Homogeneous or InHomogeneous
$5U_{n+1}+6U_n=0$	1	Homogeneous
$3U_{n+2}+U_{n+1}-2U_n=0$	2	Homogeneous
$U_{n+2}-9U_n=0$	2	Homogeneous
$U_{n+3}-5U_{n+1}+6=0$	2	InHomogeneous

Characteristic Equation

A Characteristic Equation assists us in determining an expression for **any term** whether we know its preceding terms or not.

Consider the 2nd order difference equation $U_{n+2} - 5U_{n+1} + 6U_n = 0$

We see the coefficients of each term are $\mathbf{1}U_{n+2} - \mathbf{5}U_{n+1} + \mathbf{6}U_n = 0$

Difference Equation	Homogeneous or InHomogeneous	Characteristic Equation	Roots of Equation
$U_{n+2} - 5U_{n+1} + 6U_n = 0$	Homogeneous	$1X^2 - 5X + 6 = 0$	$X=2$, $X=3$

Group Work

In groups, consider the 2nd Order homogeneous difference equations shown and determine both the characteristic equation and the roots of those equations.

$5U_{n+2} -6U_n=0$
$3U_{n+2}+ U_{n+1}-2U_n=0$
$U_{n+2} -6U_{n+1}+9U_n=0$

Feedback from Groups

Difference Equation	Characteristic Equation	Roots of Equation
$5U_{n+2} - 6U_n = 0$	$5X^2 - 6 = 0$	$X = \pm\sqrt{\frac{6}{5}}$
$3U_{n+2} + U_{n+1} - 2U_n = 0$	$3X^2 + X - 2 = 0$	$X = \frac{2}{3}, X = -1$
$U_{n+2} - 6U_{n+1} + 9U_n = 0$	$X^2 - 6X + 9 = 0$	$X = 3$

Two distinct roots A , B
$U_n = (l A^n) + (m B^n)$

Two same roots A
$U_n = (l A^n) + n(m A^n)$

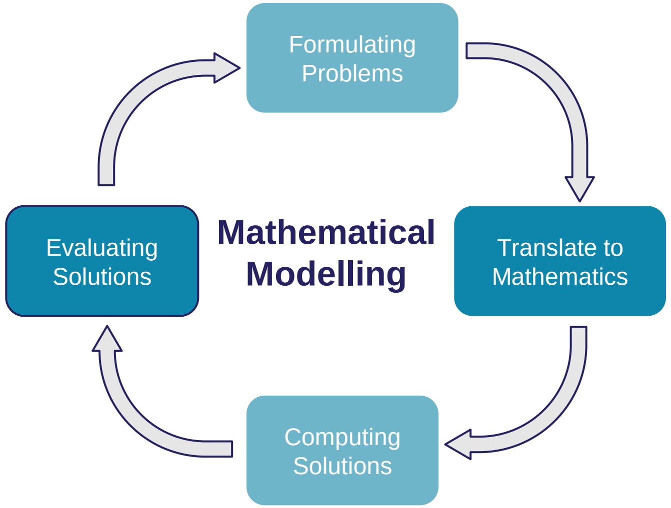
Mathematical Modelling Brief

1

Concepts, then Modelling

In many real-world settings the management of stock is an important consideration.

Choose a real-world problem related to the distribution or management of stock and model the problem(s) you have selected using *The Modelling Cycle*.



Mathematical Modelling Problem

Problem Statement:

Determine the population of trout in the river Slaney over the next few years, following the introduction of a small number of trout to the river prior to their annual breeding season.

Student-Led Enquiry

In groups,

- discuss what background research that students might consider conducting in order to bring clarity to this problem.

- consider any assumptions students may make.

Outcome of Discussion

At the start of 2021 biologists introduced **twelve trout** to an isolated area of the river just before their **annual breeding season**.

They found that the population had **doubled** by the start of 2022.

The biologists responsible assumed that the current population of trout may be modelled using a difference equation.

Determine the Population of Trout

Formulate the problem - Assumptions

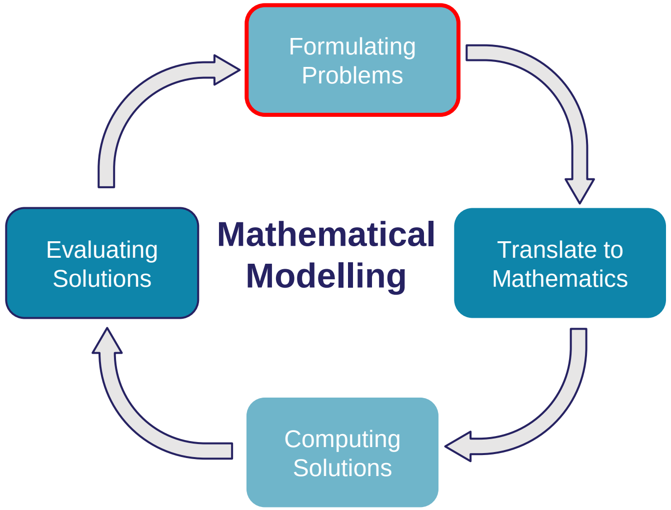
The biologist assumed that the current population of trout may be modelled by the following difference equation:

$$P_n = 2.7P_{n-1} - 1.8P_{n-2}$$

where P_n is the current population of trout in the river and $n \in N$.

$P_0 = 12$ in 2021 , $P_1 = 24$ in 2022

"Students learn that difference equations are more appropriate models If the change is discrete in time."
p.21



Determine the Population of Trout

Translate to Mathematics

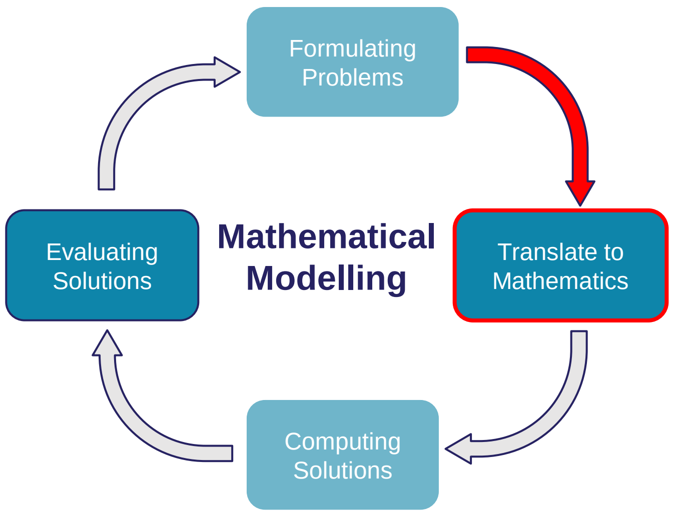
$P_n = 2.7P_{n-1} - 1.8P_{n-2}$ where P_n is the current population of trout in the river and $n \in N$.

What type of equation does this represent?

We have to

(i) Solve this difference equation.

(ii) Calculate the population of trout for say the following two years



Determine the Population of Trout

Computing the Solution

Characteristic equation $x^2 - 2.7x + 1.8 = 0$

Solving this quadratic leads to $x=3/2$ and $x=6/5$

The roots are different therefore we will $P\_n=l(\alpha^n)+m(\beta^n)$

$$P_n = l(3/2)^n + m(6/5)^n$$

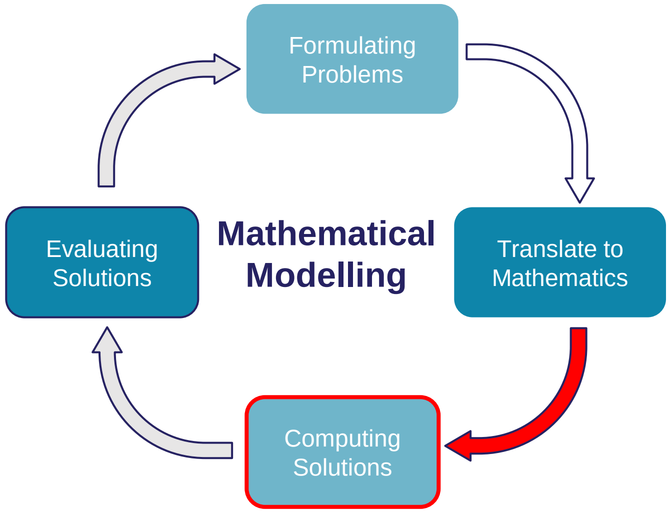
Taking $P_0=12$ and $P_1=24$ we get

$P_0=l(3/2)^0+m(6/5)^0 = 12$ ≫ $l+m=12$

$P_1=l(3/2)^1+m(6/5)^1 = 24$ ≫ $5l+4m=80$

Two distinct roots α , β

$U_n = (l\alpha^n) + (m\beta^n)$



Determine the Population of Trout

Computing the Solution

Solving these simultaneous equations

$$l + m = 12$$

$$\underline{5l + 4m = 80}$$

$$l = 32 \quad \text{and} \quad m = -20$$

$$\Rightarrow P_n = 32(3/2)^n - 20(6/5)^n$$

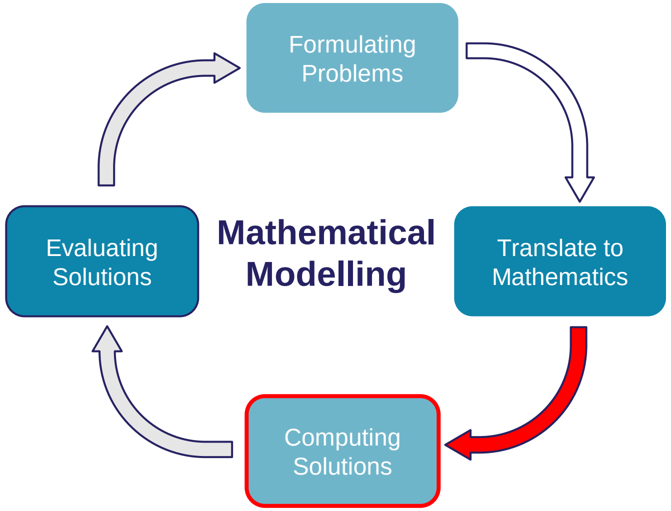
This is the solution for the difference equation.

$P_2 = 32(3/2)^2 - 20(6/5)^2 = 43.2 \approx 43$ trout in 2023

$P_3 = 32(3/2)^3 - 20(6/5)^3 = 73.44 \approx 73$ trout in 2024

The roots are different we will use equation

$$P_n = l(\alpha)^n + m(\beta)^n$$



Determine the Population of Trout

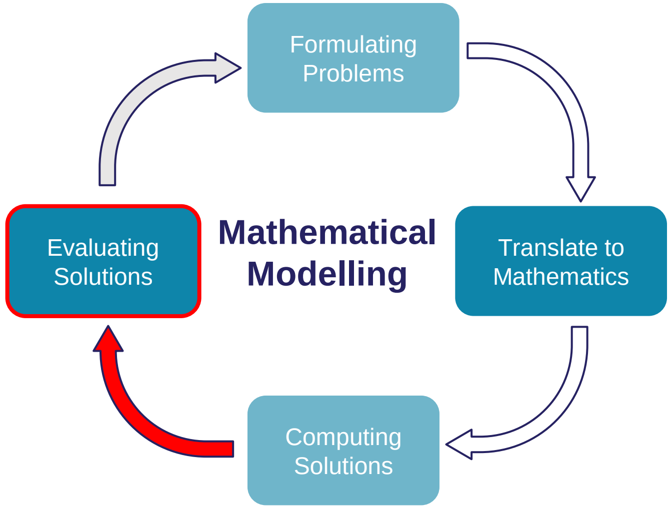
Evaluating the Solution

12 trout in 2021
24 trout in 2022
43 trout in 2023
73 trout in 2024

Does this seem accurate based on earlier assumption?

What effect would changing your variables/assumptions have on your solution?

"analyse, interpret & solve difference equations in context" *p.21*



Determine the Population of Trout

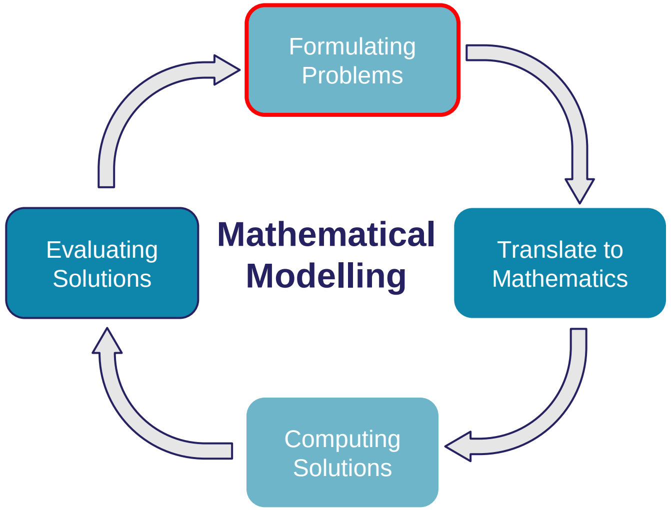
Formulate the problem - Assumptions

The biologists, flush with success, adjusted their model to factor in the redistribution of trout to other Irish rivers.

At the start of 2025 Biologists plan to **remove twenty trout** from the Slaney and rehome them in rivers throughout the country.

The biologist revised their model as follows:

$P_n = 2.7P_{n-1} – 1.8P_{n-2} - 20$ where P_n is the current population of trout in the river and $n \in N$

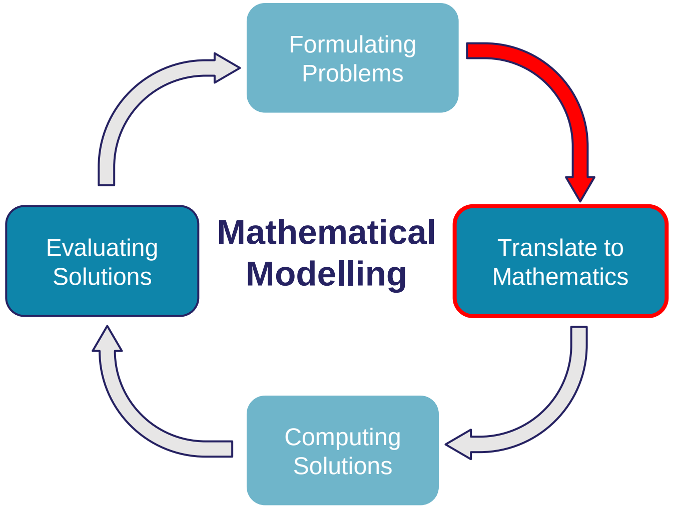


Determine the Population of Trout
Translate to Mathematics

$P_n = 2.7P_{n-1} - 1.8P_{n-2} - 20$ where P_n is the current population of trout in the river and $n \in \mathbb{N}$

What type of equation does this represent?

Using this revised model calculate the revised populations in 2025 and 2026.



Solving Inhomogeneous Equations

A **Second order inhomogeneous equation** is of the form:

$$aP_n + bP_{n-1} + cP_{n-2} = F(n)$$

F(n)	Particular solution
constant	constant a
Kn	an + b
Kn+c	an + b
Kn^2	$an^2 + bn + c$
$Kn^2 + ln + m$	$an^2 + bn + c$
kp^n	$ap^n + b$

The solution to an **inhomogeneous equation** has two components

$$P_n = [general\ soln.\ of\ associated\ homogeneous\ difference\ equation] + [particular\ soln.\ of\ full\ equation]$$

So, to solve an **inhomogeneous difference equation** we must first find the general solution to the associated equation (also known as the complimentary equation) and then the particular solution to the inhomogeneous equation.

Determine the Population of Trout
Computing the Solution

Taking P_0=43 and P_1=73

Population in 2025 P_n = 2.7(73) – 1.8(43) - 20 = 99.7 ≈ 99 trout in 2025

Population in 2026 P_n = 2.7(99) – 1.8(73) - 20 = 115.9 ≈ 115 trout in 2026

Revised modelling Equation: $P_n = 2.7P_{n-1} - 1.8P_{n-2} - 20$

Rearranging to find particular solution: $P_n - 2.7P_{n-1} + 1.8P_{n-2} = 20$

$$P_n \quad - 2.7P_{n-1} \quad + 1.8P_{n-2} \quad = 0n-20$$

$$(an+b) \quad -2.7(a(n-1)+b) \quad + 1.8(a(n-2)+b) \quad = 0n-20$$

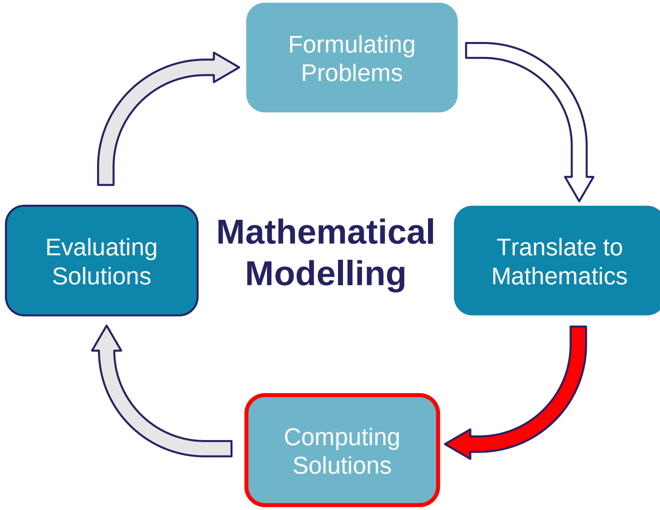
$$an+b \quad -2.7(an-a+b) \quad + 1.8(an-2a+b) \quad = 0n-20$$

$$an+b \quad -2.7an + 2.7a -2.7b + 1.8an -3.6a + 1.8b \quad = 0n-20$$

$$0.1an = 0n \Rightarrow \mathbf{a = 0}$$

$$-0.1b = 20 \Rightarrow \mathbf{b = -200}$$

F(n)	**Particular solution**
constant	constant a
Kn	an + b
Kn+c	an + b
Kn^2	$an^2 + bn + c$
$Kn^2 + ln + m$	$an^2 + bn + c$
kp^n	$ap^n + b$



Solving Inhomogeneous Equations

The solution to an **inhomogeneous equation** has two components

P_n = [general soln. of associated homogeneous difference equation] + [particular soln. of full equation]

F(n)	**Particular solution**
constant	constant a
Kn	an + b
Kn+c	an + b
Kn^2	$an^2 + bn + c$
$Kn^2 + ln + m$	$an^2 + bn + c$
kp^n	$ap^n + b$

As we found $\alpha = \frac{3}{2}$ and $\beta = \frac{6}{5}$ earlier, we can say that

$P_n = \boldsymbol{l}\left(\frac{3}{2}\right)^n + \boldsymbol{m}\left(\frac{6}{5}\right)^n$ is our general solution of associated difference equation.

We have our particular solution, $a = 0\ and\ b = 200$. So $an + b$ is represented as $\mathbf{0n - 200}$

Combining these results we have our overall general solution $\mathbf{P_n} = \boldsymbol{l}\left(\frac{3}{2}\right)^n + \boldsymbol{m}\left(\frac{6}{5}\right)^n - \mathbf{200}$

Determine the Population of Trout

Computing the Solution

$P_n = l(3/2)^n + m(6/5)^n - 200$

Using earlier estimates *2025 P_0=99* and *2026 P_1=115*

$P_n = l(3/2)^n + m(6/5)^n - 200$

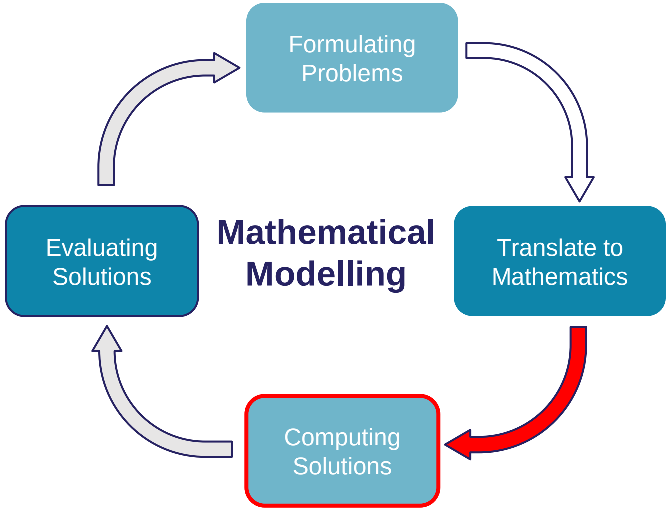
$P_0 = l(3/2)^0 + m(6/5)^0 - 200 = 99 \Rightarrow \quad l + m = 299$

$P_1 = l(3/2)^1 + m(6/5)^1 - 200 = 115 \Rightarrow \quad 15l + 12m = 3150$

Solving these simultaneous equations $l = -146$ and $m = 445$

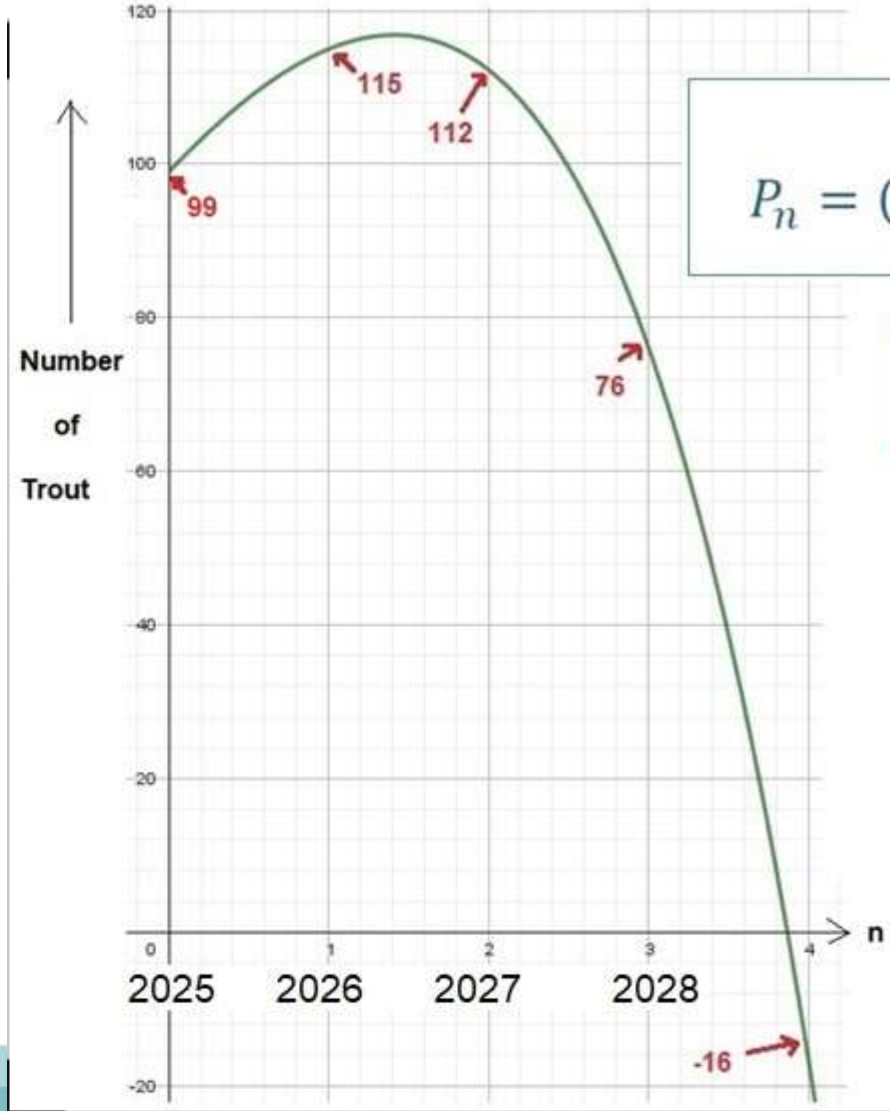
$$P_n = (-146)\left(\frac{3}{2}\right)^n + (445)\left(\frac{6}{5}\right)^n - 200$$

"solve linear and non-linear difference equations"
p.21



Determine the Population of Trout

Evaluating the Solution

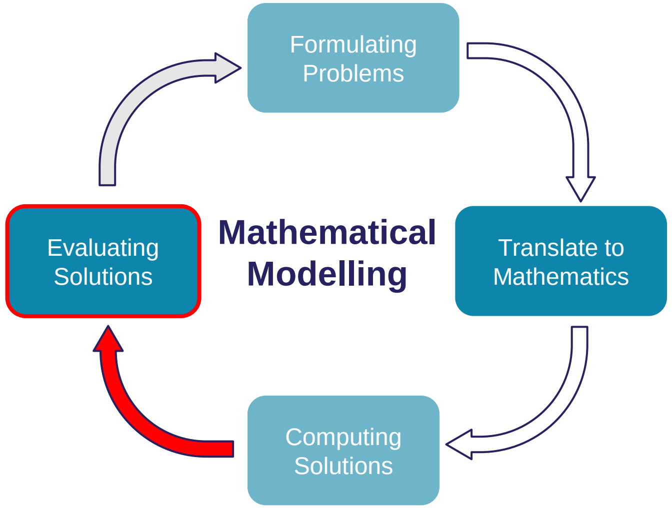


$$P_n = (-146)\left(\frac{3}{2}\right)^n + (445)\left(\frac{6}{5}\right)^n - 200$$

2025	P_0	$= 99$
2026	P_1	$= 115$
2027	P_2	$= 112$
2028	P_3	$= 76$
2029	P_4	$= -16$

[from earlier estimates]

"“analyse, interpret & solve difference equations in context”
p.21



Determine the Population of Trout

Computing the Solution

$P_n = l(3/2)^n + m(6/5)^n - 200$

However, if we used *2023 P_0=43* and *2024 P_1=73*

$P_n = l(3/2)^n + m(6/5)^n - 200$

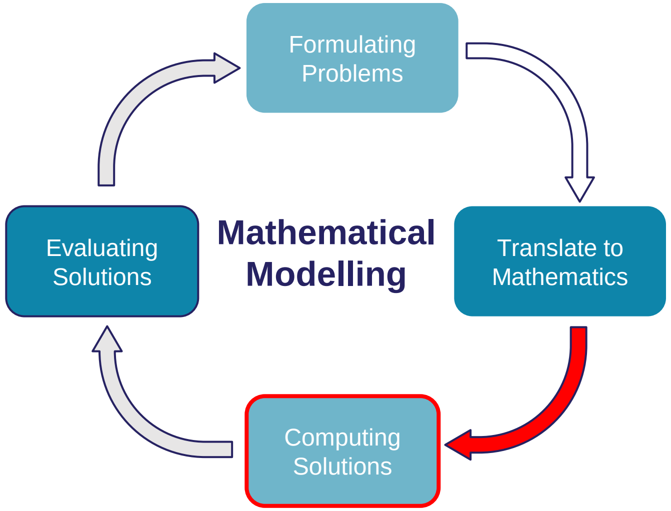
$P_0 = l(3/2)^0 + m(6/5)^0 - 200 = 43 \Rightarrow \quad l + m = 243$

$P_1 = l(3/2)^1 + m(6/5)^1 - 200 = 73 \Rightarrow \quad 15l + 12m = 2730$

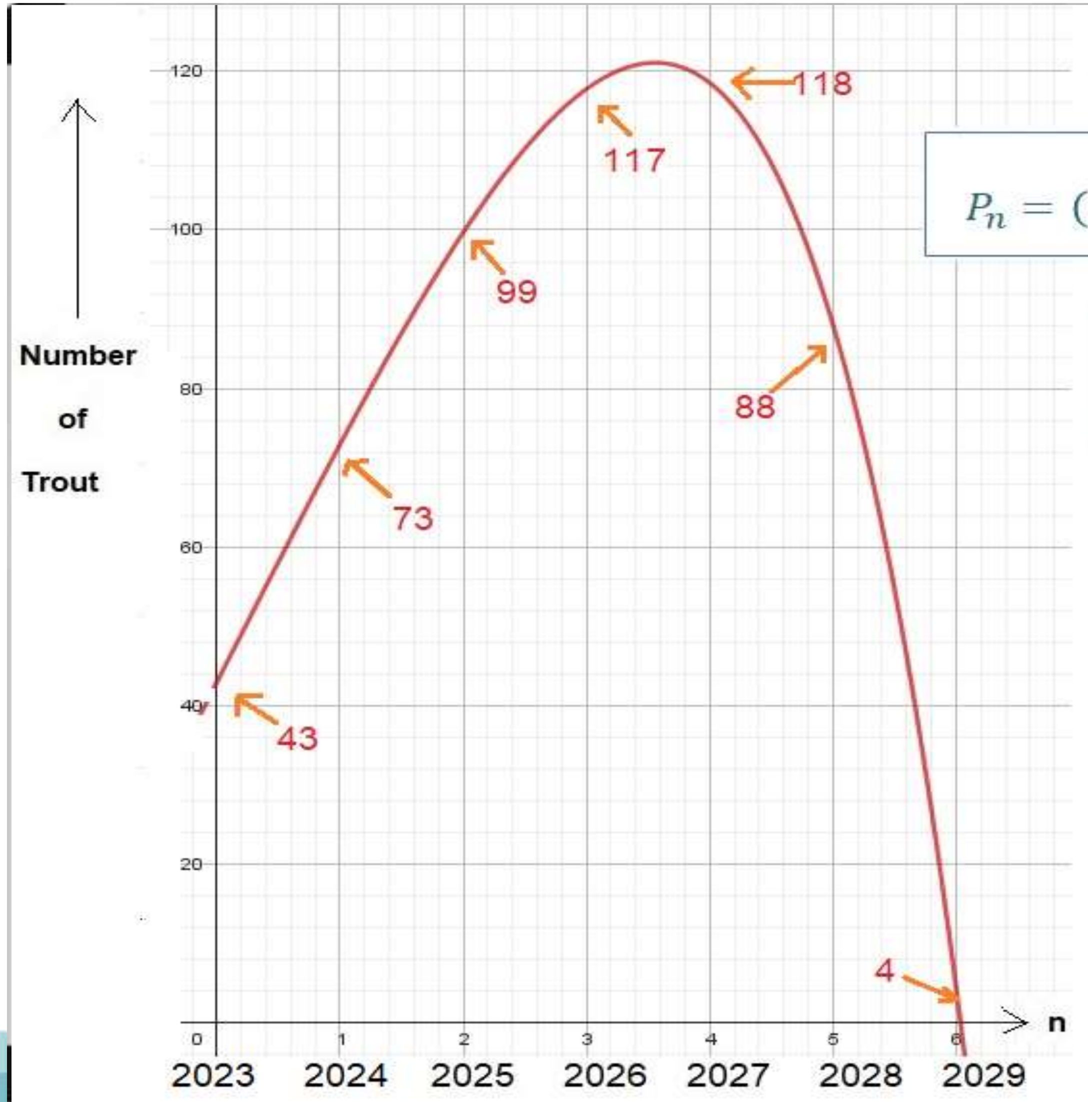
Solving these simultaneous equations $l = -62$ and $m = 305$

$$P_n = (-62)\left(\frac{3}{2}\right)^n + (305)\left(\frac{6}{5}\right)^n - 200$$

"solve linear and non-linear difference equations"
p.21



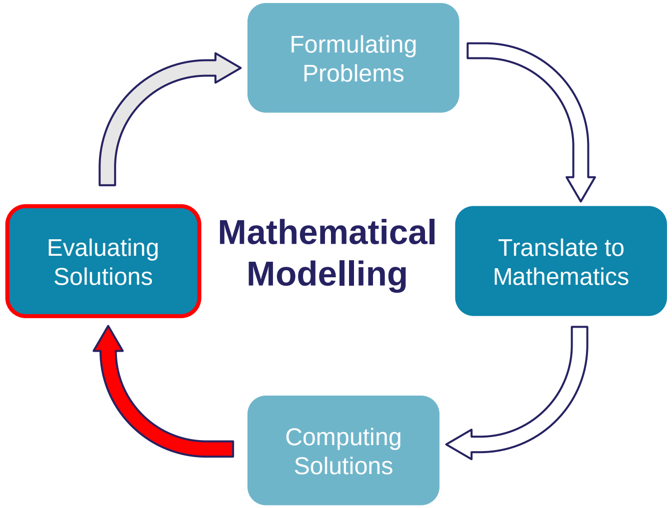
Determine the Population of Trout
Evaluating the Solution



$$P_n = (-62)\left(\frac{3}{2}\right)^n + (305)\left(\frac{6}{5}\right)^n - 200$$

2023	P_0	$= 43$
2024	P_1	$= 73$
2025	P_2	$= 99$
2026	P_3	$= 117$
2027	P_4	$= 118$

"analyse, interpret & solve difference equations in context" *p.21*



Extending The Learning

Original problem

Determine the Population of Trout in river Slaney

"..being able to critically evaluate mathematical models is a desirable skill for them to acquire" p.16

Evaluating the Solution:

How accurate and reliable is your solution based on your earlier assumptions?

What effect would changing your variables/assumptions have on your solution?

How does your solution compare with previous solutions/iterations?

Can you refine/alter your assumptions to improve your solution and will this change your solution much?

What might a further iteration look like?

How could the model be refined to improve its accuracy?

L.C. Applied Maths Mathematical Modelling: Self-Assessment Tool

Formulating the Problem:

What is the problem being asked and what research must you do?

What variables (factors) will affect your model and what assumptions will you make?

Can you predict what the output of your model will achieve and for what context (who/what) will be affected by your model?

Translating the Problem to Mathematics:

What mathematical approach will you use to solve the problem and why?

Where will your assumptions and variables be used in your model?

It is fine for a problem to have more than one solution to it depending on the assumptions chosen.

Computing the Solution:

How did you calculate your solution and what affect did your variables and assumptions have on it?

What tools (technology etc.) did you use in your solution and did this enhance your calculations?

How will you present your solution (graphs, charts, other visual aids)?

Evaluating the Solution:

How accurate is your solution based on your earlier assumptions?

Can you refine/alter your assumptions to improve your solution and will this change your solution much?

Next iteration

It is important to show how your model is improving with each iteration and why you altered your assumptions/approach.

Presenting your Final Model:

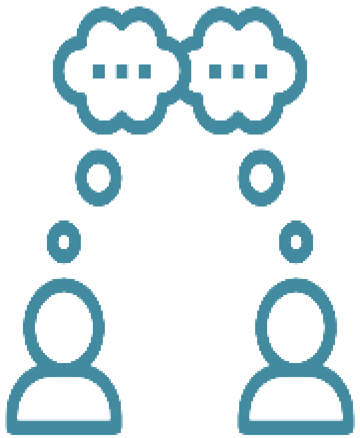
How will you present your final model so that it is well presented and easy to read?

Can visual aids be used to better communicate your work?

It may be helpful to present your work so that someone unfamiliar to your project will understand it.

Session 3: Reflection

How well did this session assist you in your understanding of how difference equations may be developed and formalised through authentic modelling problems?



How useful/relevant did you find today’s cross-curricular linking to mathematics?

Evaluation

National Seminar 9 Evaluation

https://tinyurl.com/EVALNS9

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