An Roinn Oideachais
agus Scileanna

Junior Cycle Mathematics

Contents

Introduction to junior cycle

Junior cycle education places students at the centre of the educational experience, enabling them to actively participate in their communities and in society and to be resourceful and confident learners in all aspects and stages of their lives. Junior cycle is inclusive of all students and contributes to equality of opportunity, participation and outcome for all.

The junior cycle allows students to make a greater connection with learning by focusing on the quality of learning that takes place and by offering experiences that are engaging and enjoyable for them, and relevant to their lives. These experiences are of a high quality, contribute directly to the physical, mental and social wellbeing of learners, and where possible, provide opportunities for them to develop their abilities and talents in the areas of creativity, innovation and enterprise. The learner's junior cycle programme builds on their learning to date and actively supports their progress in learning and in addition, supports them in developing the learning skills that will assist them in meeting the challenges of life beyond school.

Rationale

This mathematics specification provides students with access to important mathematical ideas to develop the mathematical knowledge and skills that they will draw on in their personal and work lives. This specification also provides students, as lifelong learners, with the basis on which further study and research in mathematics and many other fields are built.

Mathematical ideas have evolved across societies and cultures over thousands of years, and are constantly developing. Digital technologies are facilitating this expansion of ideas and provide new tools for mathematical exploration and invention. While the usefulness of mathematics for problem solving is well known, mathematics also has a fundamental role in both enabling and sustaining cultural, social, economic and technological advances and empowering individuals to become critical citizens.

The specification is underpinned by the conception of mathematics as an interconnected body of ideas and reasoning processes that students negotiate collaboratively with teachers and their peers and as independent learners. Number, measurement and geometry, statistics and probability are common aspects of most people's mathematical experiences in everyday personal, study and work situations. Equally important are the essential roles that algebra, functions and relations, logic, mathematical structure and working mathematically play in people's understanding of the natural and social worlds, and the interaction between them.

The mathematics specification builds on students' prior learning and focuses on developing increasingly sophisticated and refined mathematical understanding, fluency, reasoning, computational thinking and problem solving. These capabilities enable students to respond to familiar and unfamiliar situations by employing mathematics to make informed decisions and solve problems efficiently.

The specification supports student learning across the whole educational system by ensuring that the links between the various components of mathematics, as well as the relationship between mathematics and other subjects, are emphasised. Mathematics is composed of multiple but interrelated and interdependent concepts and structures which students can apply beyond the mathematics classroom. For example, in science, understanding sources of error and their impact on the confidence of conclusions is vital; in geography, interpretation of data underpins the study of human populations and their physical environments; in history, students need to be able to imagine timelines and time frames to reconcile related events; and in English, deriving quantitative, logical and spatial information is an important aspect of making meaning of texts. Thus the understanding of mathematics developed through study at junior cycle can inform and support students' learning across the whole educational system.

Aim

The aim of junior cycle mathematics is to provide relevant and challenging opportunities for all students to become mathematically proficient so that they can cope with the mathematical challenges of daily life and enable them to continue their study of mathematics in senior cycle and beyond. In this specification, mathematical proficiency is conceptualised not as a one-dimensional trait but as having five interconnected and interwoven components:

- conceptual understanding—comprehension of mathematical concepts, operations, and relations
- procedural fluency—skill in carrying out procedures flexibly, accurately, efficiently, and appropriately
- strategic competence—ability to formulate, represent, and solve mathematical problems in both familiar and unfamiliar contexts
- adaptive reasoning—capacity for logical thought, reflection, explanation, justification and communication
- productive disposition—habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence, perseverance and one's own efficacy.

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Overview: Links

Mathematics supports a broad range of learning experiences at junior cycle. Table 1 shows how junior cycle mathematics is linked to central features of learning and teaching in junior cycle.

Table 1: Links between junior cycle mathematics and the statements of learning

STATEMENTS OF LEARNING

The statement	Examples of possible relevant learning
SOL 1: The student communicates effectively using a variety of means in a range of contexts in L1.	Students organise, consolidate and communicate numerical and mathematical thinking clearly and coherently to peers, teachers and others verbally, and in written form using diagrams, graphs, tables and mathematical symbols.
SOL 14: The student makes informed financial decisions and develops good consumer skills.	Students learn to develop their critical thinking and reasoning skills by making value-for-money calculations and judgements which will enable them to make informed financial decisions.
SOL 15: The student recognises the potential uses of mathematical knowledge, skills and understanding in all areas of learning.	Students apply their mathematical knowledge and skills to a wide variety of problems across different subjects, including gathering, analysing, and presenting data, and using mathematics to model real-world situations.
SOL 16: The student describes, illustrates, interprets, predicts and explains patterns and relationships.	Students develop techniques to explore and understand patterns and relationships in both mathematical and non-mathematical contexts.
SOL 17: The student devises and evaluates strategies for investigating and solving problems using mathematical knowledge, reasoning and skills.	Students develop problem-solving strategies through engaging in tasks for which the solution is not immediately obvious. They reflect on their own solution strategies to such tasks and compare them to those of others as part of a collaborative learning cycle.
SOL 18: The student observes and evaluates empirical events and processes and draws valid deductions and conclusions.	Students generate and summarise data, select appropriate graphical or numerical methods to describe it, and draw conclusions from graphical and numerical summaries of the data. As part of their understanding of mathematical proof they come to appreciate the distinction between contingent deductions from particular cases, and deductions which can be proved to be universally true.
SOL 24: The student uses technology and digital media tools to learn, communicate, work and think collaboratively and creatively in a responsible and ethical manner.	Students engage with digital technology to analyse and display data numerically and graphically; to display and explore algebraic functions and their graphs; to explore shapes and solids; to investigate geometric results in a dynamic way; and to communicate and collaborate with others.

Key Skills

In addition to their specific content and knowledge, the subjects and short courses of junior cycle provide students with opportunities to develop a range of key skills. There are opportunities to support all key skills in this course but some are particularly significant.

The junior cycle curriculum focuses on eight key skills:

Figure 1: Key skills of junior cycle

KEY SKILL ELEMENTS RELATING TO MATHEMATICS

The examples below identify some of the elements that are related to learning activities in mathematics. Teachers can also build many of the other elements of key skills into their classroom planning. The eight key skills are set out in detail in Key Skills of Junior Cycle.

Table 2: Links between junior cycle mathematics and key skills

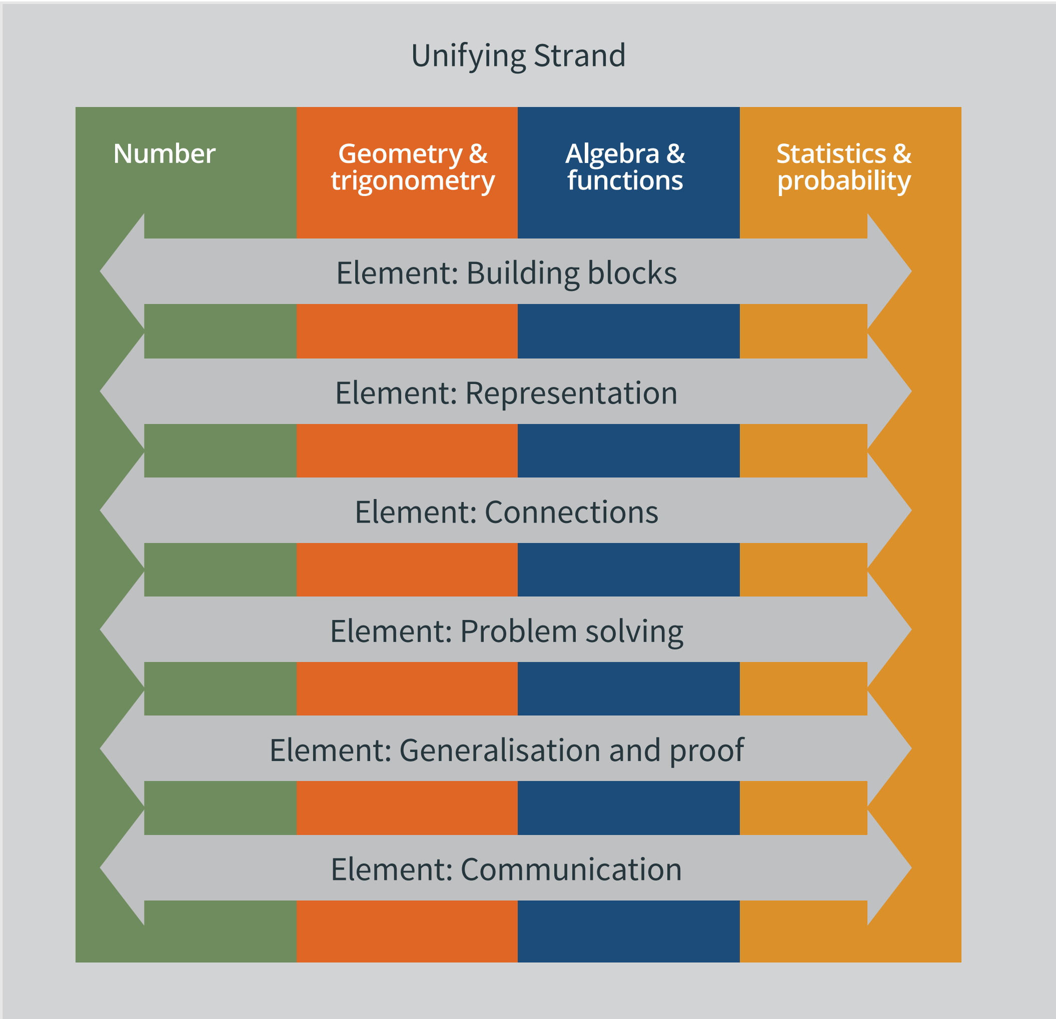
Key skill	Key skill element	Examples of possible student learning activities
Being creative	Exploring options and alternatives	As students engage in a task for which the solution is not immediately obvious, they ask questions, explore ideas and alternatives, evaluate ideas and actions and take more responsibility for their learning.
Being literate	Expressing ideas clearly and accurately	Students explain their thinking and justify their reasoning, using mathematical terminology appropriately and accurately.
Being numerate	Using digital technology to develop numeracy skills and understanding	Students use digital technology to analyse and display data numerically and graphically; to display and explore algebraic functions and their graphs; to explore shapes and solids; to investigate geometric results in a dynamic way; and to communicate and collaborate with others.
Communicating	Using numbers	Students use numbers to describe or summarise a situation; to support their reasoning and conclusions; and to convey and explain patterns and relationships.
Managing information and thinking	Thinking creatively and critically	Students engage in rich tasks which require them to use their mathematical knowledge and skills in novel ways. They reflect on their own approaches to such tasks and compare them to those of others, evaluating the strengths and weaknesses of different possible approaches.
Managing myself	Being able to reflect on my own learning	Students reflect on which learning activities they find most effective, using this knowledge to help further their learning in mathematics.
Staying well	Being confident	Students enjoy frequent opportunities to experience success in mathematics. They experience a positive approach to learning in which different approaches are valued and they are encouraged to learn from mistakes.
Working with others	Learning with others	Students work on collaborative tasks with peers in which they develop both their mathematical and their interpersonal skills, offering mutual support and feedback throughout the process.

Overview: Course

The specification for junior cycle mathematics focuses on developing students' ability to think logically, strategically, critically, and creatively through the **Unifying strand** and the four **contextual strands: Number; Geometry and trigonometry; Algebra and functions;** and **Statistics and probability.**

The specification has been designed for a minimum of 240 hours timetabled student engagement across the three years of junior cycle. This is a minimum and schools should be aware that there are students who would benefit from an engagement of more than than 240 hours to realise the national improvement targets set out in the Literacy and Numeracy strategy (DES,2011).

Figure 2: The structure of the specification for junior cycle mathematics



Unifying strand

This strand permeates all of the contextual strands and is composed of the six elements of the specification, which are shown below.

There is no specific content linked to this strand; rather, its learning outcomes underpin the rest of the specification. Each learning outcome in this strand is applicable to all of the activities and content of the other four strands—for example, students should be able to draw on all of their mathematical knowledge and skills to solve a problem or to communicate mathematics.

Furthermore, the elements of this strand are interdependent, so that students should develop the different skills associated with each element in tandem rather than in isolation – for example, engaging in problem-solving can help students improve their understanding of building blocks and their ability to make connections within mathematics.

The elements

Elements	
Building blocks	Students should understand and recall the concepts that underpin each strand, and be able to carry out the resulting procedures accurately, effectively, and appropriately.
Representation	Students should be able to represent a mathematical situation in a variety of different ways and translate flexibly between them.
Connections	Students should be able to make connections within strands and between strands, as well as connections between mathematics and the real world.
Problem solving	Students should be able to investigate patterns, formulate conjectures, and engage in tasks in which the solution is not immediately obvious, in familiar and unfamiliar contexts.
Generalisation and proof	Students should be able to move from specific instances to general mathematical statements, and to present and evaluate mathematical arguments and proofs.
Communication	Students should be able to communicate mathematics effectively in verbal and written form.

Number

This strand focuses on different aspects of number, laying the groundwork for the transition from arithmetic to algebra. Learners explore different representations of numbers and the connections between them, as well as the properties and relationships of binary operations. They investigate number patterns, and use ratio and proportionality to solve a variety of problems in numerous contexts. Learners are expected to be able to use calculators appropriately and accurately, as well as to carry out calculations by hand and mentally. They appreciate when it is appropriate to use estimation and approximation, including to check the reasonableness of results.

Geometry and trigonometry

This strand focuses on analysing characteristics and properties of two- and three-dimensional geometric shapes. Learners use geometry and trigonometry to model and solve problems involving area, length, volume, and angle measure. They develop mathematical arguments about geometric relationships and explore the concept of formal proof, using deduction to establish the validity of certain geometric conjectures and critiquing the arguments of others.

Algebra and functions

This strand focuses on representing and analysing patterns and relationships found in numbers. Building on their work in the Number strand, learners generalise their observations, expressing, interpreting, and justifying general mathematical statements in words and in symbolic notation. They use the idea of equality to form and interpret equations, and the syntactic rules of algebra to transform expressions and solve equations. Learners explore and analyse the relationships between tables, diagrams, graphs, words, and algebraic expressions as representations of functions.

Statistics and probability

This strand focuses on determining probability from random events and generating and investigating data. Students explore the relationship between experimental and theoretical probability as well as completing a data investigation; from formulating a question and designing the investigation through to interpreting their results in context and communicating their findings. Learners use graphical and numerical tools, including summary statistics and the concepts and processes of probability, to explore and analyse patterns in data. Through these activities, learners gain an understanding of data anaysis as a tool for learning about the world.

Progression from early childhood to senior cycle

EARLY CHILDHOOD

Aistear, the early childhood curriculum framework, celebrates early childhood as a time of wellbeing and enjoyment where children learn from experiences as they unfold. Children's interests and play should be the source of their first mathematical experiences. These experiences can become mathematical as they are represented and explored. Young children represent their ideas by talking, but also through models and graphics. From the motoric and sing-song beginnings of rhymes and geometric patterns built from unit blocks stem the gradual generalisation and abstraction of patterns throughout the child's day.

PRIMARY SCHOOL

The mathematics curriculum at primary school aims to provide children with a language and a system through which to analyse, describe, illustrate and explain a wide range of experiences, make predictions, and solve problems. Mathematics education seeks to enable learners to think and communicate quantitatively and spatially, solve problems, recognise situations where mathematics can be applied, and use appropriate technology to support such applications. The junior cycle mathematics specification consolidates and develops students' learning from primary school and as such experience of the learning outcomes in the Primary School Mathematics Curriculum is assumed.

SENIOR CYCLE

The junior cycle mathematics specification is developed to align with Leaving Certificate Mathematics to allow for the effective transfer of knowledge, understanding, and skills from junior to senior cycle. While certain aspects of the strands have been adapted to specifically suit junior cycle—for example, having four rather than five strands—it is nonetheless clear from the structure of this specification how students' learning in junior cycle mathematics should be developed in senior cycle. A good understanding of the knowledge and skills outlined in this specification will lay the foundations for successful engagement with senior cycle mathematics.

Expectations for Students

Expectations for students is an umbrella term that links learning outcomes with annotated examples of student work in the subject specification. When teachers, students or parents looking at the online specification scroll over the learning outcomes, a link will sometimes be available to examples of work associated with a specific learning outcome or with a group of learning outcomes. The examples of student work will have been selected to illustrate expectations and will have been annotated by teachers and will be made available alongside this specification. The examples will include work that is:

- exceptional
- above expectations
- in line with expectations.

The purpose of the examples of student work is to show the extent to which the learning outcomes are being realised in actual cases.

Learning outcomes

Learning outcomes are statements that describe what knowledge, understanding, skills and values students should be able to demonstrate having studied mathematics in junior cycle. Junior cycle mathematics is offered at Ordinary and Higher level. The majority of the learning outcomes set out in the following tables apply to all students. Additional learning outcomes for those students who take the Higher-level mathematics examination are highlighted in bold. As set out here the learning outcomes represent outcomes for students at the end of their three years of study. The specification stresses that the learning outcomes are for three years and therefore the learning outcomes focused on at a point in time will not have been 'completed', but will continue to support students' learning of mathematics up to the end of junior cycle.

The outcomes are numbered within each strand. The numbering is intended to support teacher planning in the first instance and does not imply any hierarchy of importance across the outcomes themselves. The examples of student work linked to learning outcomes will offer commentary and insights that support different standards of student work.

Unifying strand

Elements	Students should be able to:	
Building blocks	U.1	recall and demonstrate understanding of the fundamental concepts and procedures that underpin each strand
	U.2	apply the procedures associated with each strand accurately, effectively, and appropriately
	U.3.	recognise that equality is a relationship in which two mathematical expressions have the same value
Representation	U.4	represent a mathematical situation in a variety of different ways, including: numerically, algebraically, graphically, physically, in words; and to interpret, analyse, and compare such representations
Connections	U.5	make connections within and between strands
	U.6	make connections between mathematics and the real world
Problem solving	U.7	make sense of a given problem, and if necessary mathematise a situation
	U.8	apply their knowledge and skills to solve a problem, including decomposing it into manageable parts and/or simplifying it using appropriate assumptions
	U.9	interpret their solution to a problem in terms of the original question
	U.10	evaluate different possible solutions to a problem, including evaluating the reasonableness of the solutions, and exploring possible improvements and/or limitations of the solutions (if any)
Generalisation and proof	U.11	generate general mathematical statements or conjectures based on specific instances
	U.12	generate and evaluate mathematical arguments and proofs
Communication	U.13	communicate mathematics effectively: justify their reasoning, interpret their results, explain their conclusions, and use the language and notation of mathematics to express mathematical ideas precisely

Number strand

Students should be able to:

N.1 investigate the representation of numbers and arithmetic operations so that they can:

a. represent the operations of addition, subtraction, multiplication, and division in $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$ using models including the number line, decomposition, and accumulating groups of equal size

b. perform the operations of addition, subtraction, multiplication, and division and understand the relationship between these operations and the properties: commutative, associative and distributive in $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$ **and in $\mathbb{R}\backslash\mathbb{Q}$, including operating on surds**

c. explore numbers written as a^b (in index form) so that they can:

I. flexibly translate between whole numbers and index representation of numbers

II. use and apply generalisations such as $a^p a^q = a^{p+q}$; $(a^p)/(a^q) = a^{p-q}$; $(a^p)^q = a^{pq}$; and $n^{1/2} = \sqrt{n}$, for $a \in \mathbb{Z}$, and $p, q, p-q, \sqrt{n} \in \mathbb{N}$
and for $a, b, \sqrt{n} \in \mathbb{R}$, and $p, q \in \mathbb{Q}$ at *HL*

III. use and apply generalisations such as $a^0 = 1$; $a^{p/q} = \sqrt[q]{a^p} = (\sqrt[q]{a})^p$; $a^{-r} = 1/(a^r)$; $(ab)^r = a^r b^r$; and $(a/b)^r = (a^r)/(b^r)$, for $a, b \in \mathbb{R}$; $p, q \in \mathbb{Z}$; and $r \in \mathbb{Q}$

IV. generalise numerical relationships involving operations involving numbers written in index form

V. correctly use the order of arithmetic and index operations including the use of brackets

d. calculate and interpret factors (including the highest common factor), multiples (including the lowest common multiple), and prime numbers

e. present numerical answers to the degree of accuracy specified, for example, correct to the nearest hundred, to two decimal places, or to three significant figures

f. convert the number p in decimal form to the form $a \times 10^n$, where $1 \leq a < 10$, $n \in \mathbb{Z}$, $p \in \mathbb{Q}$, and $p \geq 1$ **and $0 < p < 1$**

N.2 investigate equivalent representations of rational numbers so that they can:

a. flexibly convert between fractions, decimals, and percentages

b. use and understand ratio and proportion

c. solve money-related problems including those involving bills, VAT, profit or loss, % profit or loss (on the cost price), cost price, selling price, compound interest for not more than 3 years, income tax (standard rate only), net pay (including other deductions of specified amounts), value for money calculations and judgements, **mark up (profit as a % of cost price), margin (profit as a % of selling price), compound interest, income tax and net pay (including other deductions)**

N.3 investigate situations involving proportionality so that they can:

a. use absolute and relative comparison where appropriate

b. solve problems involving proportionality including those involving currency conversion and those involving average speed, distance, and time

N.4 analyse numerical patterns in different ways, including making out tables and graphs, and continue such patterns

N.5 explore the concept of a set so that they can:

a. understand the concept of a set as a well-defined collection of elements, and that set equality is a relationship where two sets have the same elements

b. define sets by listing their elements, if finite (including in a 2-set or **3-set** Venn diagram), or by generating rules that define them

c. use and understand suitable set notation and terminology, including null set, $\varnothing$, subset, $\subset$, complement, element, $\in$, universal set, cardinal number, #, intersection, $\cap$, union, $\cup$, set difference, $\setminus$, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{R}\setminus\mathbb{Q}$

d. perform the operations of intersection and union on 2 sets **and on 3 sets**, set difference, and complement, including the use of brackets to define the order of operations

e. investigate whether the set operations of intersection, union, and difference are commutative and/or associative

Geometry and trigonometry strand

Students should be able to:

GT.1 calculate, interpret, and apply units of measure and time

GT.2 investigate 2D shapes and 3D solids so that they can:

a. draw and interpret scaled diagrams

b. draw and interpret nets of rectangular solids, **prisms (polygonal bases), cylinders**

c. find the perimeter and area of plane figures made from combinations of discs, triangles, and rectangles, including relevant operations involving pi

d. find the volume of rectangular solids, cylinders, **triangular-based prisms, spheres,** and combinations of these, including relevant operations involving pi

e. find the surface area and **curved surface area (as appropriate)** of rectangular solids, **cylinders, triangular-based prisms, spheres,** and combinations of these

GT.3 investigate the concept of proof through their engagement with geometry so that they can:

a. perform constructions 1 to 15 in *Geometry for Post-Primary School Mathematics* **(constructions 3 and 7 at HL only)**

b. recall and use the concepts, axioms, theorems, corollaries and converses, specified in *Geometry for Post-Primary School Mathematics* (section 9 for OL **and section 10 for HL)**

I. axioms 1, 2, 3, 4 and 5

II. theorems 1, 2, 3, 4, 5, 6, 9, 10, 13, 14, 15 **and 11, 12, 19**, and appropriate converses, including relevant operations involving square roots

III. corollaries 3, 4 **and 1, 2, 5** and appropriate converses

c. use **and explain** the terms: theorem, proof, axiom, corollary, converse, and implies

d. create and evaluate proofs of geometrical propositions

e. display understanding of the proofs of theorems 1, 2, 3, 4, 5, 6, 9, 10, 14, 15, **and 13, 19**; and of corollaries 3, 4, **and 1, 2, 5** (full formal proofs are not examinable)

GT.4 evaluate and use trigonometric ratios (sin, cos, and tan, defined in terms of right-angled triangles) and their inverses, involving angles between 0° and 90° at integer values **and in decimal form**

GT.5 investigate properties of points, lines and line segments in the co-ordinate plane so that they can:

a. find and interpret: distance, midpoint, slope, point of intersection, and slopes of parallel **and perpendicular** lines

b. draw graphs of line segments and interpret such graphs in context, including discussing the rate of change (slope) and the y intercept

c. find and interpret the equation of a line in the form $y = mx + c$; $y - y_1 = m(x - x_1)$; **and** $\mathbf{ax + by + c} = 0$ (for $a, b, c, m, x_1, y_1 \in Q$); including finding the slope, the y intercept, and other points on the line

GT.6 investigate transformations of simple objects so that they can:

a. recognise and draw the image of points and objects under translation, central symmetry, axial symmetry, and rotation

b. draw the axes of symmetry in shapes

Algebra and functions strand

Students should be able to:

AF.1 investigate patterns and relationships (linear, quadratic, doubling and tripling) in number, spatial patterns and real-world phenomena involving change so that they can:

a. represent these patterns and relationships in tables and graphs

b. generate a generalised expression for linear **and quadratic** patterns in words and algebraic expressions and fluently convert between each representation

c. categorise patterns as linear, non-linear, **quadratic, and exponential (doubling and tripling)** using their defining characteristics as they appear in the different representations

AF.2 investigate situations in which letters stand for quantities that are variable so that they can:

a. generate and interpret expressions in which letters stand for numbers

b. find the value of expressions given the value of the variables

c. use the concept of equality to generate and interpret equations

AF.3 apply the properties of arithmetic operations and factorisation to generate equivalent expressions so that they can develop and use appropriate strategies to:

a. add, subtract and simplify

I. linear expressions in one or more variables with coefficients in Q

II. quadratic expressions in one variable with coefficients in Z

III. expressions of the form $\frac{a}{(bx + c)}$ **, where** $a, b, c \in Z$

b. multiply expressions of the form

I. $a(bx + cy + d)$; $a(bx^2 + cx + d)$; and $ax(bx^2 + cx + d)$, where $a, b, c, d \in Z$

II. $(ax + b)(cx + d)$ and **$(ax + b)(cx^2 + dx + e)$,** where $a, b, c, d, e \in Z$

c. divide quadratic **and cubic expressions** by linear expressions, where all coefficients are integers and there is no remainder

d. flexibly convert between the factorised and expanded forms of algebraic expressions of the form:

I. axy, where $a \in Z$

II. $axy + byz$, where $a, b \in Z$

III. $sx - ty + tx - sy$, where $s, t \in Z$

IV. $dx^2 + bx$; $x^2 + bx + c$; **and $ax^2 + bx + c$,** where $b, c, d \in Z$ **and $a \in N$**

V. $x^2 - a^2$ **and $a^2x^2 - b^2y^2$,** where $a, b \in Z$

AF.4 select and use suitable strategies (graphic, numeric, algebraic, trial and improvement, working backwards) for finding solutions to:

a. linear equations in one variable with coefficients in $\mathbb{Q}$ and solutions in $\mathbb{Z}$ **or in $\mathbb{Q}$**

b. quadratic equations in one variable with coefficients and solutions in Z
coefficients in $\mathbb{Q}$ and solutions in R

c. simultaneous linear equations in two variables with coefficients and solutions in $\mathbb{Z}$
or in $\mathbb{Q}$

d. linear inequalities in one variable of the form $g(x) < k$, and graph the solution sets on the number line for $x \in \mathbb{N}$, $\mathbb{Z}$, and $\mathbb{R}$

AF.5 generate quadratic equations given integer roots

AF.6 apply the relationship between operations and an understanding of the order of operations including brackets and exponents to change the subject of a formula

AF.7 investigate functions so that they can:

a. demonstrate understanding of the concept of a function

b. represent and interpret functions in different ways—graphically (for $x \in \mathbb{N}$, $\mathbb{Z}$, and $\mathbb{R}$, [continuous functions only], as appropriate), diagrammatically, in words, and algebraically — using the language and notation of functions (domain, range, co-domain, $f(x) =$, $f : x \longmapsto$, and $y =$) (drawing the graph of a function given its algebraic expression is limited to linear and quadratic functions at *OL*)

c. use graphical methods to find and interpret approximate solutions of equations such as $f(x) = g(x)$
and approximate solution sets of inequalities such as $f(x) < g(x)$

d. make connections between the shape of a graph and the story of a phenomenon, including identifying and interpreting maximum and minimum points

Statistics and probability strand

Students should be able to:

SP.1 investigate the outcomes of experiments so that they can:

a. generate a sample space for an experiment in a systematic way, including tree diagrams for successive events and two-way tables for independent events

b. use the fundamental principle of counting to solve authentic problems

SP.2 investigate random events so that they can:

a. demonstrate understanding that probability is a measure on a scale of 0-1 of how likely an event (including an everyday event) is to occur

b. use the principle that, in the case of equally likely outcomes, the probability of an event is given by the number of outcomes of interest divided by the total number of outcomes

c. use relative frequency as an estimate of the probability of an event, given experimental data, and recognise that increasing the number of times an experiment is repeated generally leads to progressively better estimates of its theoretical probability

SP.3 carry out a statistical investigation which includes the ability to:

a. generate a statistical question

b. plan and implement a method to generate and/or source unbiased, representative data, and present this data in a frequency table

c. classify data (categorical, numerical)

d. select, draw and interpret appropriate graphical displays of univariate data, including pie charts, bar charts, line plots, histograms (equal intervals), ordered stem and leaf plots, **and ordered back-to-back stem and leaf plots**

e. select, calculate and interpret appropriate summary statistics to describe aspects of univariate data. Central tendency: mean **(including of a grouped frequency distribution),** median, mode. Variability: range

f. evaluate the effectiveness of different graphical displays in representing data

g. discuss misconceptions and misuses of statistics

h. discuss the assumptions and limitations of conclusions drawn from sample data or graphical/numerical summaries of data

Assessment and Reporting

Assessment in education involves gathering, interpreting and using information about the processes and outcomes of learning. It takes different forms and can be used in a variety of ways, such as to record and report achievement, to determine appropriate routes for learners to take through a differentiated curriculum, or to identify specific areas of difficulty or strength for a given learner. While different techniques may be employed for formative, diagnostic and summative purposes, the focus of the assessment and reporting is on the improvement of student learning. To do this it must fully reflect the aim of the curriculum.

The junior cycle places a strong emphasis on assessment as part of the learning process. This approach requires a more varied approach to assessment in ensuring that the assessment method or methods chosen are fit for purpose, timely and relevant to students. Assessment in junior cycle mathematics will optimise the opportunity for students to become reflective and active participants in their learning and for teachers to support this. This rests upon the provision for learners of opportunities to negotiate success criteria against which the quality of their work can be judged by peer, self, and teacher assessment; and upon the quality of the focused feedback they get in support of their learning.

Providing focused feedback to students on their learning is a critical component of high-quality assessment and a key factor in building students' capacity to manage their own learning and their motivation to stick with a complex task or problem. Assessment is most effective when it moves beyond marks and grades, and reporting focuses not just on how the student has done in the past but on the next steps for further learning. This approach will ensure that assessment takes place as close as possible to the point of learning. Summative assessment still has a role to play, but is only one element of a broader approach to assessment.

Essentially, the purpose of assessment and reporting at this stage of education is to support learning. Parents/guardians should receive a comprehensive picture of student learning. Linking classroom assessment and other assessment with a new system of reporting that culminates in the awarding of the Junior Cycle Profile of Achievement (JCPA) will offer parents/guardians a clear and broad picture of their child's learning journey over the three years of junior cycle.

To support this, teachers and schools will have access to an Assessment Toolkit. Along with the guide to the Subject Learning and Assessment Review (SLAR) process, the Assessment Toolkit will include learning, teaching and assessment support material, including:

- formative assessment
- planning for and designing assessment
- ongoing assessments for classroom use
- judging student work – looking at expectations for students and features of quality
- reporting to parents and students
- thinking about assessment: ideas, research and reflections
- a glossary.

The contents of the Assessment Toolkit will include a range of assessment supports, advice and guidelines that will enable schools and teachers to engage with the new assessment system and reporting arrangements in an informed way, with confidence and clarity.

Assessment for the Junior Cycle Profile of Achievement

The assessment of mathematics for the purposes of the Junior Cycle Profile of Achievement (JCPA) will comprise two Classroom-Based Assessments: CBA 1; and CBA 2. In addition, the second Classroom-Based Assessment will have a written Assessment Task that will be marked, along with a final examination, by the State Examinations Commission.

Rationale for the Classroom-Based Assessments in Mathematics

Over the three years of junior cycle, students will be provided with many opportunities to enjoy and learn mathematics. The Classroom-Based Assessments, outlined overleaf, link to the priorities for learning and teaching in mathematics, with a particular emphasis on problem solving and communicating. Through the Classroom-Based Assessments students will develop and demonstrate their mathematical proficiency by actively engaging in practical and authentic learning experiences.

The Classroom-Based Assessments will be carried out by all students, and will be marked at a common level. The teacher's judgement of their mathematical attainment will be recorded for subject learning and assessment review, as well as for the school's reporting to parents and students.

Classroom-Based Assessment 1:

CBA	Format	Student preparation	Completion of assessment	SLAR meeting
Mathematical investigation	Report which may be presented in a wide range of formats	Students will, over a three-week period, follow the *Problem-solving cycle* to investigate a mathematical problem. **Problem-solving cycle:** define a problem; decompose it into manageable parts and/or simplify it using appropriate assumptions; translate the problem to mathematics if necessary; engage with the problem and solve it if possible; interpret any findings in the context of the original problem.	End of second year	One review meeting

Classroom-Based Assessment 2:

CBA	Format	Student preparation	Completion of assessment	SLAR meeting
Statistical investigation	Report which may be presented in a wide range of formats	Students will, over a three-week period; follow the *Statistical enquiry cycle*. **Statistical enquiry cycle:** formulate a question; plan and collect unbiased, representative data; organise and manage the data; explore and analyse the data using appropriate displays and numerical summaries and answer the original question giving reasons based on the analysis section.	End of first term of third year	One review meeting

Assessing the Classroom-Based Assessments

More detailed material on assessment for reporting in junior cycle mathematics, setting out details of the practical arrangements related to assessment of the Classroom-Based Assessments, will be available in separate Assessment Guidelines for Mathematics. This will include, for example, the suggested length and formats for student pieces of work, and support in using 'on balance' judgement in relation to the features of quality.

The NCCA's Assessment Toolkit will also include substantial resource material for use and reference in the ongoing classroom assessment of junior cycle mathematics, as well as providing a detailed account of the Subject Learning and Assessment Review process.

Features of quality

The features of quality support student and teacher judgement of the Classroom-Based Assessments and are the criteria that will be used by teachers to assess the pieces of student work. All students will complete both CBAs. The features of quality will be available in Assessment Guidelines for Mathematics.

Assessment Task

The Assessment Task is a written task completed by students during class time, which is not marked by the class teacher, but is sent to the State Examinations Commission for marking. It will be allocated 10% of the marks used to determine the grade awarded by the SEC. The Assessment Task is specified by the NCCA and is related to the learning outcomes on which the second Classroom-Based Assessment is based. The content and format of the Assessment Task may vary from year to year.

FINAL EXAMINATION

There will be two examination papers, one at Ordinary and one at Higher level, set and marked by the State Examinations Commission (SEC). The examination will be two hours in duration and will take place in June of third year. The number of questions on the examination papers may vary from year to year. In any year, the learning outcomes to be assessed will constitute a sample of the relevant outcomes from the tables of learning outcomes.

INCLUSIVE ASSESSMENT PRACTICES

This specification allows for inclusive assessment practices whether as part of ongoing assessment or Classroom-Based Assessments. Where a school judges that a student has a specific physical or learning difficulty, reasonable accommodations may be put in place to remove, as far as possible, the impact of the disability on the student's performance in Classroom-Based Assessments. The accommodations, e.g. the support provided by a special needs assistant or the support of assistive technologies, should be in line with the arrangements the school has put in place to support the student's learning throughout the year.

Appendix A: Glossary of action verbs

This glossary is designed to clarify the learning outcomes. Each action verb is described in terms of what the learner should be able to do once they have achieved the learning outcome. This glossary will be aligned with the command words used in the assessment.

Action verbs	Students should be able to
Analyse	study or examine something in detail, break down to bring out the essential elements or structure; identify parts and relationships, and to interpret information to reach conclusions
Apply	select and use knowledge and/or skills to solve a problem in a new situation
Calculate	work out a numerical answer
Classify	group things based on common characteristics
Compare	give an account of the similarities and (or) differences between two (or more) items or situations, referring to both (all) of them throughout
Construct	use properties of shapes and geometric results to draw accurately, using only the prescribed geometrical tools
Convert	change from one form to another
Define	[a set]: give a rule that identifies the elements of a set
Discuss	offer a considered, balanced review that includes a range of arguments, factors or hypotheses; opinions or conclusions should be presented clearly and supported by appropriate evidence
Estimate	state or calculate a rough value for a particular quantity
Evaluate	judge the relative quality or validity of something, which may include analysing, comparing and contrasting, criticising, defending, or judging
Explain	give a reasoned account, showing how causes lead to outcomes
Generalise	generate a general statement based on specific instances
Generate	produce or create
Interpret	use knowledge and understanding to explain the meaning of something in context
Investigate	observe, study, or make a detailed and systematic examination to establish facts and reach new conclusions
Justify	give valid reasons or evidence to support an answer or conclusion
Mathematise	generate a mathematical representation (e.g. graph, equation, geometric figure) to describe a particular aspect of a phenomenon

Action verbs	Students should be able to
Prove	give a deductive argument to demonstrate that a particular statement is true, including reasons for each step in the argument
Round	give the number in the required form (for example, a multiple of 100, or a number with three significant figures) that is closest in absolute terms to a particular number
Sketch	draw a rough diagram or graph without using geometrical tools
Solve	work out an answer or solution to
State	provide a concise statement with little or no supporting argument
Understand	have detailed knowledge of, be able to use appropriately, and see the connections between parts
Use	apply knowledge or rules to put theory into practice
Verify	demonstrate that a statement is true

Appendix B: Geometry for Post-primary School Mathematics

At a glance: Definitions, Axioms, Theorems and Corollaries

Page Number	
41	Axiom 4. Congruent triangles. Definition 20. Right angled triangle. Hypotenuse Definition 21. Isosceles triangle. Equilateral. Scalene Theorem 2. In an isosceles triangle, the angles opposite the equal sides are equal. Converse Theorem 2. If two angles are equal, then the tringle is isosceles.
42	Definition 22. Parallel lines. Axiom 5. Axiom of parallels. Definition 23. Transversal line. Definition 24. Alternate angles.
43	Theorem 3. If a transversal makes equal alternate angles on two lines, then the lines are parallel. Converse of Theorem 3. If two lines are parallel, then any transversal will make equal alternate angles with them.
44	Theorem 4. The angles in any triangle add to 180 degrees.
45	Definition 25. Corresponding angles. Theorem 5. Two lines are parallel if and only if for any transversal, corresponding angles are equal.
46	Definition 26. Exterior angle. Interior opposite angles. Theorem 6. Each exterior angle of a triangle is equal to the sum of the interior opposite angles.
47	Theorem 7. In a triangle, the angle opposite the greater of two sides is greater than the angle opposite the lesser side. Converse of Theorem 7. The side opposite the greater of two angles is greater than the side opposite the lesser angle
48	Theorem 8. Two sides of a triangle are together greater than the third.
49	Definition 27. Perpendicular bisector. Definition 28. Polygon, sides, vertices, adjacent sides, adjacent vertices, adjacent angles. Definition 29. Quadrilateral, opposite side, opposite angles.
50	Definition 30. Rectangle. Definition 31. Rhombus. Definition 32. Square. Definition 33. Polygon, equilateral and regular. Definition 34. Parallelogram. Theorem 9. In a parallelogram, opposite sides are equal, and opposite angles are equal.

Page Number	
51	Converse 1 of theorem 9. If the opposite angles of a convex quadrilateral are equal, then it is a parallelogram. Converse 2 of theorem 9. If the opposite sides of a convex quadrilateral are equal, then it is a parallelogram. Corollary 1. A diagonal divides a parallelogram into two congruent triangles. Theorem 10. The diagonals of a parallelogram bisect one another.
52	Definition 35. Similar Triangles. Theorem 11. If three parallel lines cut off equal segments on some transversal line, then they will cut off equal segments on any other transversal.
53	Definition 36. Division of a line segment in a given ratio. Theorem 12. In triangle *ABC*, If a line l is parallel to *BC* and cuts *[AB]* in the ratio s:t, then it cuts *[AC]* in the same ratio.
55	Theorem 13. If two triangles are similar then their sides are proportional in order.
56	Theorem 14. Pythagoras.
57	Theorem 15. Converse to Pythagoras. If the square of one side of a triangle is the sum of the squares of the other two then the angle opposite the first side is a right angle.
58	Definition 37. Apex, height and altitude of a triangle. Theorem 16. For a triangle, base times height does not depend on the choice of base.
59	Definition 38. Area of a triangle
60	Theorem 17. A diagonal of a parallelogram bisects the area. Definition 39. Height and base of a parallelogram.
61	Theorem 18. The area of a parallelogram is the base by the height. Definition 40. Circle, its centre, radius, diameter, sector, circumference, semi-circle, disc, chord, arc, standing on an arc, standing on a chord. Theorem 19. The angle at the centre of a circle standing on a given arc is twice the angle at any point of the circle standing on the same arc.
62	Corollary 2. All angles at points of the circle, standing on the same arc, are equal.
63	Corollary 3. Each angle in a semi-circle is a right angle. Corollary 4. If the angle standing on a chord at the same point of a circle is a right angle, then the chord is a diameter. Definition 41. Cyclic quadrilateral. Corollary 5. In a cyclic quadrilateral opposite angles sum to 180°.

1 Introduction

The Junior Certificate and Leaving Certificate mathematics course committees of the National Council for Curriculum and Assessment (NCCA) accepted the recommendation contained in the paper [4] to base the logical structure of post-primary school geometry on the level 1 account in Professor Barry's book [1].

To quote from [4]: We distinguish three levels:

Level 1: The fully-rigorous level, likely to be intelligible only to professional mathematicians and advanced third- and fourth-level students.

Level 2: The semiformal level, suitable for digestion by many students from (roughly) the age of 14 and upwards.

Level 3: The informal level, suitable for younger children.

This document sets out the agreed geometry for post-primary schools. It has been updated to reflect the changes to the study of geometry introduced in the 2018 Junior Cycle Mathematics specification. Readers should refer to the Junior Cycle Mathematics specification and the Leaving Certificate Mathematics syllabus document for the expected learning outcomes at each level. A summary of the corresponding content is given in sections 9–13 of this document.

2 The system of geometry used for the purposes of formal proofs

In the following, Geometry refers to plane geometry.

There are many formal presentations of geometry in existence, each with its own set of axioms and primitive concepts. What constitutes a valid proof in the context of one system might therefore not be valid in the context of another. Given that students will be expected to present formal proofs in the examinations, it is therefore necessary to specify the system of geometry that is to form the context for such proofs.

The formal underpinning for the system of geometry on the Junior and Leaving Certificate courses is that described by Prof. Patrick D. Barry in [1]. A properly formal presentation of such a system has the serious disadvantage that it is not readily accessible to students at this level. Accordingly, what is presented below is a necessarily simplified version that treats many concepts far more loosely than a truly formal presentation would demand. Any readers who wish to rectify this deficiency are referred to [1] for a proper scholarly treatment of the material.

Barry's system has the primitive undefined terms **plane, point, line, $<_l$ (precedes on a line), (open) half-plane, distance, and degree-measure**, and seven axioms: A_1: about incidence, A_2: about order on lines, A_3: about how lines separate the plane, A_4: about distance, A_5: about degree measure, A_6: about congruence of triangles, A_7: about parallels.

3 Guiding Principles

In constructing a level 2 account, we respect the principles about the relationship between the levels laid down in [4, Section 2].

The choice of material to study should be guided by applications (inside and outside Mathematics proper).

The most important reason to study synthetic geometry is to prepare the ground logically for the development of trigonometry, coordinate geometry, and vectors, which in turn have myriad applications.

We aim to keep the account as simple as possible.

We also take it as desirable that the official Irish syllabus should avoid imposing terminology that is nonstandard in international practice, or is used in a nonstandard way.

No proof should be allowed at level 2 that cannot be expanded to a complete rigorous proof at level 1, or that uses axioms or theorems that come later in the logical sequence. We aim to supply adequate proofs for all the theorems, but do not propose that only those proofs will be acceptable. It should be open to teachers and students to think about other ways to prove the results, provided they are correct and fit within the logical framework. Indeed, such activity is to be encouraged. Naturally, teachers and students will need some assurance that such variant proofs will be acceptable if presented in examination. We suggest that the discoverer of a new proof should discuss it with students and colleagues, and (if in any doubt) should refer it to the National Council for Curriculum and Assessment and/or the State Examinations Commission.

It may be helpful to note the following non-exhaustive list of salient differences between Barry's treatment and our less formal presentation.

- Whereas we may use set notation and we expect students to understand the conceptualisation of geometry in terms of sets, we more often use the language that is common when discussing geometry informally, such as "the point is/lies on the line", "the line passes through the point", etc.

- We accept and use a much lesser degree of precision in language and notation (as is apparent from some of the other items on this list).

- We state five explicit axioms, employing more informal language than Barry's, and we do not explicitly state axioms corresponding to Axioms A2 and A3 – instead we make statements without fanfare in the text.

- We accept a much looser understanding of what constitutes an **angle**, making no reference to angle-supports. We do not define the term angle. We mention reflex angles from the beginning (but make no use of them until we come to angles in circles), and quietly assume (when the time comes) that axioms that are presented by Barry in the context of wedge-angles apply also in the naturally corresponding way to reflex angles.

- When naming an angle, it is always assumed that the non-reflex angle is being referred to, unless the word "reflex" precedes or follows.

- We make no reference to results such as Pasch's property and the "crossbar theorem". (That is, we do not expect students to consider the necessity to prove such results or to have them given as axioms.)

- We refer to "the number of degrees" in an angle, whereas Barry treats this more correctly as "the degree-measure" of an angle.

- We take it that the definitions of parallelism, perpendicularity and "sidedness" are readily extended from lines to half-lines and line segments. (Hence, for example, we may refer to the opposite sides of a particular quadrilateral as being parallel, meaning that the lines of which they are subsets are parallel).

- We do not refer explicitly to triangles being **congruent** "under the correspondence $(A, B, C) \to (D, E, F)$", taking it instead that the correspondence is the one implied by the order in which the vertices are listed. That is, when we say "ΔABC is congruent to ΔDEF" we mean, using Barry's terminology, "Triangle [A,B,C] is congruent to triangle [D,E,F] under the correspondence $(A, B, C) \to (D, E, F)$".

- We do not always retain the distinction in language between an angle and its measure, relying frequently instead on the context to make the meaning clear. However, we continue the practice of distinguishing notationally between the angle $\angle ABC$ and the number $|\angle ABC|$ of degrees in the angle[1]. In the same spirit, we may refer to two angles being equal, or one being equal to the sum of two others, (when we should more precisely say that the two are equal in measure, or that the measure of one is equal to the sum of the measures of the other two). Similarly, with length, we may loosely say, for example: "opposite sides of a parallelogram are equal", or refer to "a circle of radius r". Where ambiguity does not arise, we may refer to angles using a single letter. That is, for example, if a diagram includes only two rays or segments from the point A, then the angle concerned may be referred to as $\angle A$.

Having pointed out these differences, it is perhaps worth mentioning some significant structural aspects of Barry's geometry that are retained in our less formal version:

[1] In practice, the examiners do not penalise students who leave out the bars.

- The primitive terms are almost the same, subject to the fact that their properties are conceived less formally. We treat **angle** as an extra undefined term.

- We assume that results are established in the same order as in Barry [1], up to minor local rearrangement. The exception to this is that we state all the axioms as soon as they are useful, and we bring the theorem on the angle-sum in a triangle forward to the earliest possible point (short of making it an axiom). This simplifies the proofs of a few theorems, at the expense of making it easy to see which results are theorems of so-called Neutral Geometry[2].

- **Area** is not taken to be a primitive term or a given property of regions. Rather, it is defined for triangles following the establishment of the requisite result that the products of the lengths of the sides of a triangle with their corresponding altitudes are equal, and then extended to convex quadrilaterals.

- **Isometries or other transformations** are not taken as primitive. Indeed, in our case, the treatment does not extend as far as defining them. Thus they can play no role in our proofs.

4 Outline of the Level 2 Account

We present the account by outlining:

1. A list (Section 5), of the terminology for the geometrical concepts. Each term in a theory is either undefined or defined, or at least definable. There have to be some undefined terms. (In textbooks, the undefined terms will be introduced by descriptions, and some of the defined terms will be given explicit definitions, in language appropriate to the level. We assume that previous level 3 work will have laid a foundation that will allow students to understand the undefined terms. We do not give the explicit definitions of all the definable terms. Instead we rely on the student's ordinary language, supplemented sometimes by informal remarks. For instance, we do not write out in cold blood the definition of the **side opposite** a given angle in a triangle, or the

[2] Geometry without the axiom of parallels. This is not a concern in secondary school.

definition (in terms of set membership) of what it means to say that a line **passes through** a given point. The reason why some terms **must** be given explicit definitions is that there are alternatives, and the definition specifies the starting point; the alternative descriptions of the term are then obtained as theorems.

2. A logical account (Section 6) of the synthetic geometry theory. All the material through to LC higher is presented. The individual syllabuses will identify the relevant content by referencing it by number (e.g. Theorems 1,2, 9).

3. The geometrical constructions (Section 7) that will be studied. Again, the individual syllabuses will refer to the items on this list by number when specifying what is to be studied.

4. Some guidance on teaching (Section 8).

5. Syllabus content summaries for each of JC-OL, JC-HL, LC-FL, LC-OL, LC-HL.

5 Terms

Undefined Terms: angle, degree, length, line, plane, point, ray, real number, set.

Most important Defined Terms: area, parallel lines, parallelogram, right angle, triangle, congruent triangles, similar triangles, tangent to a circle, area.

Other Defined terms: acute angle, alternate angles, angle bisector, arc, area of a disc, base and corresponding apex and height of triangle or parallelogram, chord, circle, circumcentre, circumcircle, circumference of a circle, circumradius, collinear points, concurrent lines, convex quadrilateral, corresponding angles, diameter, disc, distance, equilateral triangle, exterior angles of a triangle, full angle, hypotenuse, incentre, incircle, inradius, interior opposite angles, isosceles triangle, median lines, midpoint of a segment, null angle, obtuse angle, perpendicular bisector of a segment, perpendicular lines, point of contact of a tangent, polygon, quadrilateral, radius, ratio, rectangle, reflex angle ordinary angle, rhombus, right-angled triangle, scalene triangle, sector, segment, square, straight angle, subset, supplementary angles, transversal line, vertically-opposite angles.

Definable terms used without explicit definition: angles, adjacent sides, arms or sides of an angle, centre of a circle, endpoints of segment, equal angles, equal segments, line passes through point, opposite sides or angles of a quadrilateral, or vertices of triangles or quadrilaterals, point lies on line, side of a line, side of a polygon, the side opposite an angle of a triangle, vertex, vertices (of angle, triangle, polygon).

6 The Theory

Line[3] is short for straight line. Take a fixed **plane**[4], once and for all, and consider just lines that lie in it. The plane and the lines are **sets**[5] of **points**[6]. Each line is a **subset** of the plane, i.e. each element of a line is a point of the plane. Each line is endless, extending forever in both directions. Each line has infinitely-many points. The points on a line can be taken to be ordered along the line in a natural way. As a consequence, given any three distinct points on a line, exactly one of them lies **between** the other two. Points that are not on a given line can be said to be on one or other **side** of the line. The sides of a line are sometimes referred to as **half-planes**.

Notation 1. We denote points by roman capital letters A, B, C, etc., and lines by lower-case roman letters l, m, n, etc.

Axioms are statements we will accept as true[7].

Axiom 1 (Two Points Axiom)**.** *There is exactly one line through any two given points. (We denote the line through A and B by AB.)*

Definition 1. The line **segment** $[AB]$ is the part of the line AB between A and B (including the endpoints). The point A divides the line AB into two

[3]Line is undefined.
[4]Undefined term
[5]Undefined term
[6]Undefined term
[7] An **axiom** is a statement accepted without proof, as a basis for argument. A **theorem** is a statement deduced from the axioms by logical argument.

pieces, called **rays**. The point A lies between all points of one ray and all points of the other. We denote the ray that starts at A and passes through B by $[AB$. Rays are sometimes referred to as **half-lines**.

Three points usually determine three different lines.

Definition 2. If three or more points lie on a single line, we say they are **collinear**.

Definition 3. Let A, B and C be points that are not collinear. The **triangle** ΔABC is the piece of the plane enclosed by the three line segments $[AB]$, $[BC]$ and $[CA]$. The segments are called its **sides**, and the points are called its **vertices** (singular **vertex**).

6.1 Length and Distance

We denote the set of all **real numbers**[8] by $\mathbb{R}$.

Definition 4. We denote the **distance**[9] between the points A and B by $|AB|$. We define the **length** of the segment $[AB]$ to be $|AB|$.

We often denote the lengths of the three sides of a triangle by a, b, and c. The usual thing for a triangle ΔABC is to take $a = |BC|$, i.e. the length of the side opposite the vertex A, and similarly $b = |CA|$ and $c = |AB|$.

Axiom 2 (Ruler Axiom[10])**.** *The distance between points has the following properties:*

1. *the distance $|AB|$ is never negative;*
2. *$|AB| = |BA|$;*
3. *if C lies on AB, between A and B, then $|AB| = |AC| + |CB|$;*
4. *(marking off a distance) given any ray from A, and given any real number $k \geq 0$, there is a unique point B on the ray whose distance from A is k.*

---

[8] Undefined term

[9] Undefined term

[10] Teachers used to traditional treatments that follow Euclid closely should note that this axiom (and the later Protractor Axiom) guarantees the existence of various points (and lines) without appeal to postulates about constructions using straight-edge and compass. They are powerful axioms.

Definition 5. The **midpoint** of the segment $[AB]$ is the point M of the segment with [11]

$$|AM| = |MB| = \frac{|AB|}{2}.$$

6.2 Angles

Definition 6. A subset of the plane is **convex** if it contains the whole segment that connects any two of its points.

For example, one side of any line is a convex set, and triangles are convex sets.

We do not define the term angle formally. Instead we say: There are things called **angles**. To each angle is associated:

1. a unique point A, called its **vertex**;

2. two rays $[AB$ and $[AC$, both starting at the vertex, and called the **arms** of the angle;

3. a piece of the plane called the **inside** of the angle.

An angle is either a null angle, an ordinary angle, a straight angle, a reflex angle or a full angle, Unless otherwise specified, you may take it that any angle we talk about is an ordinary angle.

Definition 7. An angle is a **null angle** if its arms coincide with one another and its inside is the empty set.

Definition 8. An angle is an **ordinary angle** if its arms are not on one line, and its inside is a convex set.

Definition 9. An angle is a **straight angle** if its arms are the two halves of one line, and its inside is one of the sides of that line.

Definition 10. An angle is a **reflex angle** if its arms are not on one line, and its inside is not a convex set.

Definition 11. An angle is a **full angle** if its arms coincide with one another and its inside is the rest of the plane.

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[11] Students may notice that the first equality implies the second.

Definition 12. Suppose that A, B, and C are three noncollinear points. We denote the (ordinary) angle with arms $[AB$ and $[AC$ by $\angle BAC$ (and also by $\angle CAB$). We shall also use the notation $\angle BAC$ to refer to straight angles, where A, B, C are collinear, and A lies between B and C (either side could be the inside of this angle).

Sometimes we want to refer to an angle without naming points, and in that case we use lower-case Greek letters, α,β,γ, etc.

6.3 Degrees

Notation 2. We denote the number of **degrees** in an angle $\angle BAC$ or α by the symbol $|\angle BAC|$, or $|\angle \alpha|$, as the case may be.

Axiom 3 (Protractor Axiom)**.** *The number of degrees in an angle (also known as its degree-measure) is always a number between* $0°$ *and* $360°$*. The number of degrees of an ordinary angle is less than* $180°$*. It has these properties:*

1. *A straight angle has* $180°$*.*
2. *Given a ray* $[AB$*, and a number* d *between* 0 *and* 180*, there is exactly one ray from* A *on each side of the line* AB *that makes an (ordinary) angle having* d *degrees with the ray* $[AB$*.*
3. *If* D *is a point inside an angle* $\angle BAC$*, then*

$$|\angle BAC| = |\angle BAD| + |\angle DAC|.$$

Null angles are assigned $0°$, full angles $360°$, and reflex angles have more than $180°$. To be more exact, if A, B, and C are noncollinear points, then the reflex angle "outside" the angle $\angle BAC$ measures $360° - |\angle BAC|$, in degrees.

Definition 13. The ray $[AD$ is the **bisector** of the angle $\angle BAC$ if

$$|\angle BAD| = |\angle DAC| = \frac{|\angle BAC|}{2}.$$

We say that an angle is 'an angle of' (for instance) $45°$, if it has 45 degrees in it.

Definition 14. A **right angle** is an angle of exactly $90°$.

Definition 15. An angle is **acute** if it has less than $90°$, and **obtuse** if it has more than $90°$.

Definition 16. If $\angle BAC$ is a straight angle, and D is off the line BC, then $\angle BAD$ and $\angle DAC$ are called **supplementary angles**. They add to $180°$.

Definition 17. When two lines AB and AC cross at a point A, they are **perpendicular** if $\angle BAC$ is a right angle.

Definition 18. Let A lie between B and C on the line BC, and also between D and E on the line DE. Then $\angle BAD$ and $\angle CAE$ are called **vertically-opposite angles**.

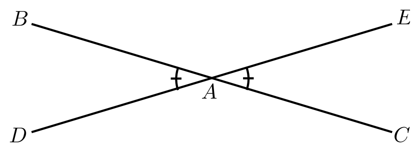

Figure 1.

Theorem 1 (Vertically-opposite Angles)**.**
Vertically opposite angles are equal in measure.

Proof. See Figure 1. The idea is to add the same supplementary angles to both, getting $180°$. In detail,

$$\begin{aligned} |\angle BAD| + |\angle BAE| &= 180°, \\ |\angle CAE| + |\angle BAE| &= 180°, \end{aligned}$$

so subtracting gives:

$$\begin{aligned} |\angle BAD| - |\angle CAE| &= 0°, \\ |\angle BAD| &= |\angle CAE|. \end{aligned}$$

□

6.4 Congruent Triangles

Definition 19. Let A, B, C and A', B', C' be triples of non-collinear points. We say that the triangles ΔABC and $\Delta A'B'C'$ are **congruent** if all the sides and angles of one are equal to the corresponding sides and angles of the other, i.e. $|AB| = |A'B'|$, $|BC| = |B'C'|$, $|CA| = |C'A'|$, $|\angle ABC| = |\angle A'B'C'|$, $|\angle BCA| = |\angle B'C'A'|$, and $|\angle CAB| = |\angle C'A'B'|$. See Figure 2.

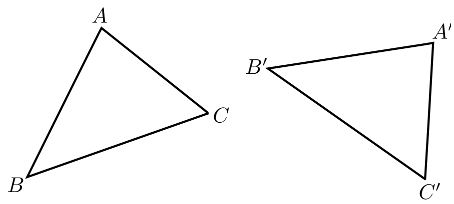

Figure 2.

Notation 3. Usually, we abbreviate the names of the angles in a triangle, by labelling them by the names of the vertices. For instance, we write $\angle A$ for $\angle CAB$.

Axiom 4 (SAS+ASA+SSS[12])**.**
If (1) $|AB| = |A'B'|$, $|AC| = |A'C'|$ *and* $|\angle A| = |\angle A'|$,
or
(2) $|BC| = |B'C'|$, $|\angle B| = |\angle B'|$, *and* $|\angle C| = |\angle C'|$,
or
(3) $|AB| = |A'B'|$, $|BC| = |B'C'|$, *and* $|CA| = |C'A'|$
then the triangles ΔABC *and* $\Delta A'B'C'$ *are congruent.*

Definition 20. A triangle is called **right-angled** if one of its angles is a right angle. The other two angles then add to $90°$, by Theorem 4, so are both acute angles. The side opposite the right angle is called the **hypotenuse**.

Definition 21. A triangle is called **isosceles** if two sides are equal[13]. It is **equilateral** if all three sides are equal. It is **scalene** if no two sides are equal.

Theorem 2 (Isosceles Triangles)**.**
(1) In an isosceles triangle the angles opposite the equal sides are equal.
(2) Conversely, If two angles are equal, then the triangle is isosceles.

Proof. (1) Suppose the triangle ΔABC has $AB = AC$ (as in Figure 3). Then ΔABC is congruent to ΔACB [SAS]
$\therefore \angle B = \angle C$.

[12]It would be possible to prove all the theorems using a weaker axiom (just SAS). We use this stronger version to shorten the course.
[13] The simple “equal” is preferred to “of equal length”

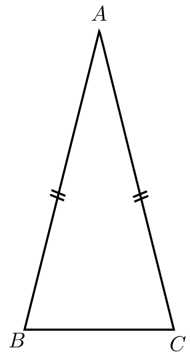

Figure 3.

(2) Suppose now that $\angle B = \angle C$. Then
ΔABC is congruent to ΔACB [ASA]
$\therefore |AB| = |AC|$, ΔABC is isosceles. □

Acceptable Alternative Proof of (1). Let D be the midpoint of $[BC]$, and use SSS to show that the triangles ΔABD and ΔACD are congruent. (This proof is more complicated, but has the advantage that it yields the extra information that the angles $\angle ADB$ and $\angle ADC$ are equal, and hence both are right angles (since they add to a straight angle)). □

6.5 Parallels

Definition 22. Two lines l and m are **parallel** if they are either identical, or have no common point.

Notation 4. We write $l||m$ for "l is parallel to m".

Axiom 5 (Axiom of Parallels)**.** *Given any line l and a point P, there is exactly one line through P that is parallel to l.*

Definition 23. If l and m are lines, then a line n is called a **transversal** of l and m if it meets them both.

Definition 24. Given two lines AB and CD and a transversal BC of them, as in Figure 4, the angles $\angle ABC$ and $\angle BCD$ are called **alternate** angles.

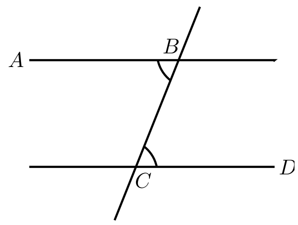

Figure 4.

Theorem 3 (Alternate Angles)**.** *Suppose that A and D are on opposite sides of the line BC.*
(1) If $|\angle ABC| = |\angle BCD|$, then $AB||CD$. In other words, if a transversal makes equal alternate angles on two lines, then the lines are parallel.
(2) Conversely, if $AB||CD$, then $|\angle ABC| = |\angle BCD|$. In other words, if two lines are parallel, then any transversal will make equal alternate angles with them.

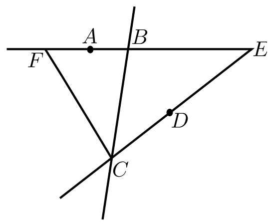

Figure 5.

Proof. (1) Suppose $|\angle ABC| = |\angle BCD|$. If the lines AB and CD do not meet, then they are parallel, by definition, and we are done. Otherwise, they meet at some point, say E. Let us assume that E is on the same side of BC as D.[14] Take F on EB, on the same side of BC as A, with $|BF| = |CE|$ (see Figure 5). [Ruler Axiom]

[14]Fuller detail: There are three cases:
1°: E lies on BC. Then (using Axiom 1) we must have $E = B = C$, and $AB = CD$.
2°: E lies on the same side of BC as D. In that case, take F on EB, on the same side of BC as A, with $|BF| = |CE|$. [Ruler Axiom]
Then ΔBCE is congruent to ΔCBF. [SAS]
Thus

$$|\angle BCF| = |\angle CBE| = 180^\circ - |\angle ABC| = 180^\circ - |\angle BCD|,$$

Then ΔBCE is congruent to ΔCBF. [SAS]
Thus

$$|\angle BCF| = |\angle CBE| = 180° - |\angle ABC| = 180° - |\angle BCD|,$$

so that F lies on DC. [Ruler Axiom]
Thus AB and CD both pass through E and F, and hence coincide, [Axiom 1]
Hence AB and CD are parallel. [Definition of parallel]

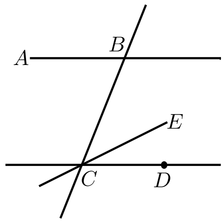

Figure 6.

(2) To prove the converse, suppose $AB||CD$. Pick a point E on the same side of BC as D with $|\angle BCE| = |\angle ABC|$. (See Figure 6.) By Part (1), the line CE is parallel to AB. By Axiom 5, there is only one line through C parallel to AB, so $CE = CD$. Thus $|\angle BCD| = |\angle BCE| = |\angle ABC|$. □

Theorem 4 (Angle Sum 180)**.** *The angles in any triangle add to* $180°$*.*

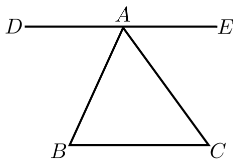

Figure 7.

---

so that F lies on DC. [Protractor Axiom]
Thus AB and CD both pass through E and F, and hence coincide. [Axiom 1]
3°: E lies on the same side of BC as A. Similar to the previous case.
Thus, in all three cases, $AB = CD$, so the lines are parallel.

Proof. Let ΔABC be given. Take a segment $[DE]$ passing through A, parallel to BC, with D on the opposite side of AB from C, and E on the opposite side of AC from B (as in Figure 7). [Axiom of Parallels]
Then AB is a transversal of DE and BC, so by the Alternate Angles Theorem,

$$|\angle ABC| = |\angle DAB|.$$

Similarly, AC is a transversal of DE and BC, so

$$|\angle ACB| = |\angle CAE|.$$

Thus, using the Protractor Axiom to add the angles,

$$\begin{aligned} & |\angle ABC| + |\angle ACB| + |\angle BAC| \\ = & |\angle DAB| + |\angle CAE| + |\angle BAC| \\ = & |\angle DAE| = 180^\circ, \end{aligned}$$

since $\angle DAE$ is a straight angle. □

Definition 25. Given two lines AB and CD, and a transversal AE of them, as in Figure 8(a), the angles $\angle EAB$ and $\angle ACD$ are called **corresponding** angles[15].

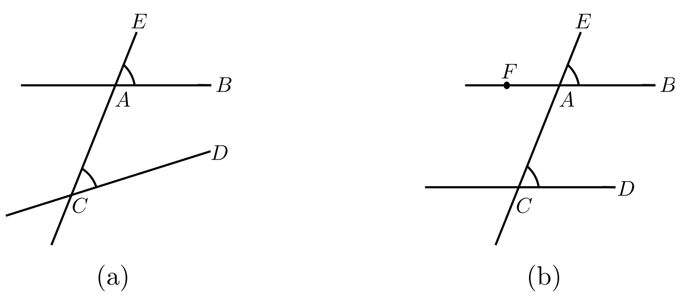

Figure 8.

Theorem 5 (Corresponding Angles)**.** *Two lines are parallel if and only if for any transversal, corresponding angles are equal.*

[15]with respect to the two lines and the given transversal.

Proof. See Figure 8(b). We first assume that the corresponding angles $\angle EAB$ and $\angle ACD$ are equal. Let F be a point on AB such that F and B are on opposite sides of AE. Then we have

$|\angle EAB| = |\angle FAC|$ [Vertically opposite angles]

Hence the alternate angles $\angle FAC$ and $\angle ACD$ are equal and therefore the lines $FA = AB$ and CD are parallel.

For the converse, let us assume that the lines AB and CD are parallel. Then the alternate angles $\angle FAC$ and $\angle ACD$ are equal. Since

$|\angle EAB| = |\angle FAC|$ [Vertically opposite angles]

we have that the corresponding angles $\angle EAB$ and $\angle ACD$ are equal. □

Definition 26. In Figure 9, the angle α is called an **exterior angle** of the triangle, and the angles β and γ are called (corresponding) **interior opposite angles**.[16]

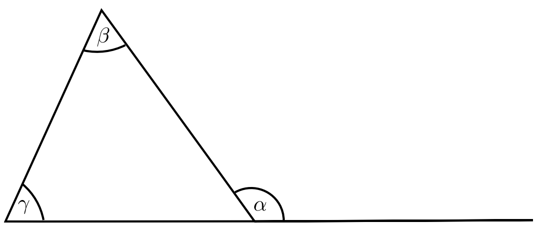

Figure 9.

Theorem 6 (Exterior Angle)**.** *Each exterior angle of a triangle is equal to the sum of the interior opposite angles.*

Proof. See Figure 10. In the triangle ΔABC let α be an exterior angle at A. Then

$|\alpha| + |\angle A| = 180°$ [Supplementary angles]

and

$|\angle B| + |\angle C| + |\angle A| = 180°$. [Angle sum 180°]

Subtracting the two equations yields $|\alpha| = |\angle B| + |\angle C|$. □

[16] The phrase **interior remote angles** is sometimes used instead of **interior opposite angles**.

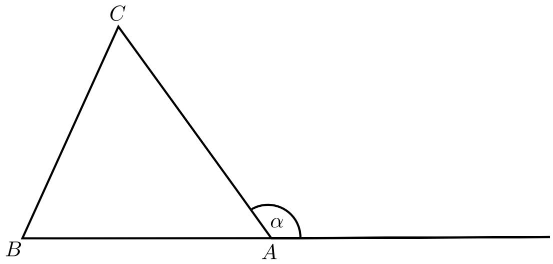

Figure 10.

Theorem 7.
(1) In ΔABC, suppose that $|AC| > |AB|$. Then $|\angle ABC| > |\angle ACB|$. In other words, the angle opposite the greater of two sides is greater than the angle opposite the lesser side.
(2) Conversely, if $|\angle ABC| > |\angle ACB|$, then $|AC| > |AB|$. In other words, the side opposite the greater of two angles is greater than the side opposite the lesser angle.

Proof.
(1) Suppose that $|AC| > |AB|$. Then take the point D on the segment $[AC]$ with
$|AD| = |AB|$. [Ruler Axiom]

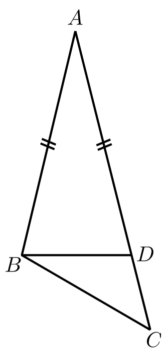

Figure 11.

See Figure 11. Then ΔABD is isosceles, so

$$\begin{aligned} |\angle ACB| &< |\angle ADB| && \text{[Exterior Angle]} \\ &= |\angle ABD| && \text{[Isosceles Triangle]} \\ &< |\angle ABC|. \end{aligned}$$

Thus $|\angle ACB| < |\angle ABC|$, as required.

(2)(This is a Proof by Contradiction!)
Suppose that $|\angle ABC| > |\angle ACB|$. See Figure 12.

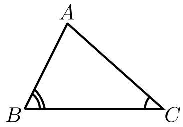

Figure 12.

If it could happen that $|AC| \leq |AB|$, then
either Case 1°: $|AC| = |AB|$, in which case ΔABC is isosceles, and then $|\angle ABC| = |\angle ACB|$, which contradicts our assumption,
or Case 2°: $|AC| < |AB|$, in which case Part (1) tells us that $|\angle ABC| < |\angle ACB|$, which also contradicts our assumption. Thus it cannot happen, and we conclude that $|AC| > |AB|$. □

Theorem 8 (Triangle Inequality)**.**
Two sides of a triangle are together greater than the third.

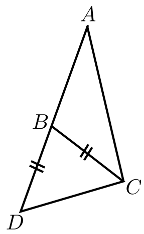

Figure 13.

Proof. Let ΔABC be an arbitrary triangle. We choose the point D on AB such that B lies in $[AD]$ and $|BD| = |BC|$ (as in Figure 13). In particular

$$|AD| = |AB| + |BD| = |AB| + |BC|.$$

Since B lies in the angle $\angle ACD$[17] we have

$$|\angle BCD| < |\angle ACD|.$$

[17] B lies in a segment whose endpoints are on the arms of $\angle ACD$. Since this angle is $< 180°$ its inside is convex.

Because of $|BD| = |BC|$ and the Theorem about Isosceles Triangles we have $|\angle BCD| = |\angle BDC|$, hence $|\angle ADC| = |\angle BDC| < |\angle ACD|$. By the previous theorem applied to ΔADC we have

$$|AC| < |AD| = |AB| + |BC|.$$

□

6.6 Perpendicular Lines

Proposition 1. [18] *Two lines perpendicular to the same line are parallel to one another.*

Proof. This is a special case of the Alternate Angles Theorem. □

Proposition 2. *There is a unique line perpendicular to a given line and passing though a given point. This applies to a point on or off the line.*

Definition 27. The **perpendicular bisector** of a segment $[AB]$ is the line through the midpoint of $[AB]$, perpendicular to AB.

6.7 Quadrilaterals and Parallelograms

Definition 28. A closed chain of line segments laid end-to-end, not crossing anywhere, and not making a straight angle at any endpoint encloses a piece of the plane called a **polygon**. The segments are called the **sides** or edges of the polygon, and the endpoints where they meet are called its **vertices**. Sides that meet are called **adjacent sides**, and the ends of a side are called **adjacent vertices**. The angles at adjacent vertices are called **adjacent angles**. A polygon is called **convex** if it contains the whole segment connecting any two of its points.

Definition 29. A **quadrilateral** is a polygon with four vertices.

Two sides of a quadrilateral that are not adjacent are called **opposite sides**. Similarly, two angles of a quadrilateral that are not adjacent are called **opposite angles**.

[18] In this document, a proposition is a useful or interesting statement that could be proved at this point, but whose proof is not stipulated as an essential part of the programme. Teachers are free to deal with them as they see fit. For instance, they might be just mentioned, or discussed without formal proof, or used to give practice in reasoning for HLC students. It is desirable that they be mentioned, at least.

Definition 30. A **rectangle** is a quadrilateral having right angles at all four vertices.

Definition 31. A **rhombus** is a quadrilateral having all four sides equal.

Definition 32. A **square** is a rectangular rhombus.

Definition 33. A polygon is **equilateral** if all its sides are equal, and **regular** if all its sides and angles are equal.

Definition 34. A **parallelogram** is a quadrilateral for which both pairs of opposite sides are parallel.

Proposition 3. *Each rectangle is a parallelogram.*

Theorem 9. *In a parallelogram, opposite sides are equal, and opposite angles are equal.*

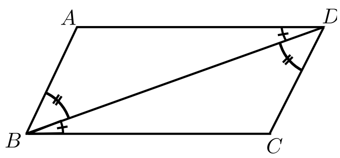

Figure 14.

Proof. See Figure 14. Idea: Use Alternate Angle Theorem, then ASA to show that a diagonal divides the parallelogram into two congruent triangles. This gives opposite sides and (one pair of) opposite angles equal.

In more detail, let $ABCD$ be a given parallelogram, $AB||CD$ and $AD||BC$. Then

$|\angle ABD| = |\angle BDC|$ [Alternate Angle Theorem]
$|\angle ADB| = |\angle DBC|$ [Alternate Angle Theorem]
ΔDAB is congruent to ΔBCD. [ASA]

$$\therefore |AB| = |CD|, \ |AD| = |CB|, \text{ and } |\angle DAB| = |\angle BCD|.$$

□

Remark 1. Sometimes it happens that the converse of a true statement is false. For example, it is true that if a quadrilateral is a rhombus, then its diagonals are perpendicular. But it is not true that a quadrilateral whose diagonals are perpendicular is always a rhombus.

It may also happen that a statement admits several valid converses. Theorem 9 has two:

Converse 1 to Theorem 9: *If the opposite angles of a convex quadrilateral are equal, then it is a parallelogram.*

Proof. First, one deduces from Theorem 4 that the angle sum in the quadrilateral is 360°. It follows that adjacent angles add to 180°. Theorem 3 then yields the result. □

Converse 2 to Theorem 9: *If the opposite sides of a convex quadrilateral are equal, then it is a parallelogram.*

Proof. Drawing a diagonal, and using SSS, one sees that opposite angles are equal. □

Corollary 1. *A diagonal divides a parallelogram into two congruent triangles.*

Remark 2. The converse is false: It may happen that a diagonal divides a convex quadrilateral into two congruent triangles, even though the quadrilateral is not a parallelogram.

Proposition 4. *A quadrilateral in which one pair of opposite sides is equal and parallel, is a parallelogram.*

Proposition 5. *Each rhombus is a parallelogram.*

Theorem 10. *The diagonals of a parallelogram bisect one another.*

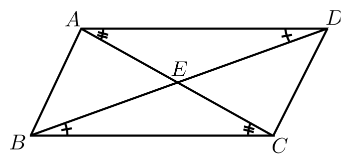

Figure 15.

Proof. See Figure 15. Idea: Use Alternate Angles and ASA to establish congruence of ΔADE and ΔCBE.

In detail: Let AC cut BD in E. Then

$$\begin{aligned} |\angle EAD| &= |\angle ECB| \text{ and} \\ |\angle EDA| &= |\angle EBC| && \text{[Alternate Angle Theorem]} \\ |AD| &= |BC|. && \text{[Theorem 9]} \end{aligned}$$

$\therefore \Delta ADE$ is congruent to ΔCBE. [ASA] □

Proposition 6 (Converse)**.** *If the diagonals of a quadrilateral bisect one another, then the quadrilateral is a parallelogram.*

Proof. Use SAS and Vertically Opposite Angles to establish congruence of ΔABE and ΔCDE. Then use Alternate Angles. □

6.8 Ratios and Similarity

Definition 35. If the three angles of one triangle are equal, respectively, to those of another, then the two triangles are said to be **similar**.

Remark 3. Obviously, two right-angled triangles are similar if they have a common angle other than the right angle.

(The angles sum to 180°, so the third angles must agree as well.)

Theorem 11. *If three parallel lines cut off equal segments on some transversal line, then they will cut off equal segments on any other transversal.*

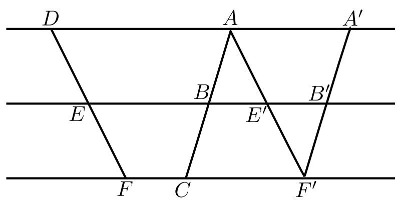

Figure 16.

Proof. Uses opposite sides of a parallelogram, AAS, Axiom of Parallels.

In more detail, suppose $AD||BE||CF$ and $|AB| = |BC|$. We wish to show that $|DE| = |EF|$.

Draw $AE'||DE$, cutting EB at E' and CF at F'.
Draw $F'B'||AB$, cutting EB at B'. See Figure 16.

Then

$\|B'F'\|$	$=$	$\|BC\|$	[Theorem 9]
	$=$	$\|AB\|$.	[by Assumption]
$\|\angle BAE'\|$	$=$	$\|\angle E'F'B'\|$.	[Alternate Angle Theorem]
$\|\angle AE'B\|$	$=$	$\|\angle F'E'B'\|$.	[Vertically Opposite Angles]
$\therefore \Delta ABE'$	is congruent to	$\Delta F'B'E'$.	[ASA]
$\therefore \|AE'\|$	$=$	$\|F'E'\|$.	

But
$|AE'| = |DE|$ and $|F'E'| = |FE|$. [Theorem 9]
$\therefore |DE| = |EF|$. □

Definition 36. Let s and t be positive real numbers. We say that a point C **divides the segment** $[AB]$ **in the ratio** $s : t$ if C lies on the line AB, and is between A and B, and

$$\frac{|AC|}{|CB|} = \frac{s}{t}.$$

We say that a line l **cuts** $[AB]$ **in the ratio** $s : t$ if it meets AB at a point C that divides $[AB]$ in the ratio $s : t$.

Remark 4. It follows from the Ruler Axiom that given two points A and B, and a ratio $s : t$, there is exactly one point that divides the segment $[AB]$ in that exact ratio.

Theorem 12. *Let ΔABC be a triangle. If a line l is parallel to BC and cuts $[AB]$ in the ratio $s : t$, then it also cuts $[AC]$ in the same ratio.*

Proof. We prove only the commensurable case.

Let l cut $[AB]$ in D in the ratio $m : n$ with natural numbers m, n. Thus there are points (Figure 17)

$$D_0 = A, D_1, D_2, \dots, D_{m-1}, D_m = D, D_{m+1}, \dots, D_{m+n-1}, D_{m+n} = B,$$

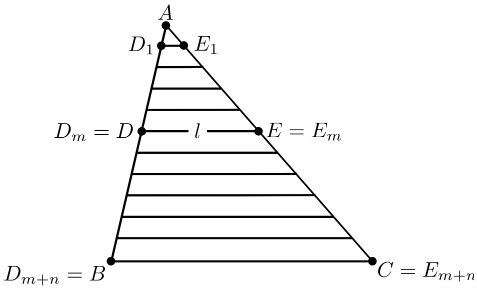

Figure 17.

equally spaced along $[AB]$, i.e. the segments

$$[D_0D_1], [D_1D_2], \ldots [D_iD_{i+1}], \ldots [D_{m+n-1}D_{m+n}]$$

have equal length.

Draw lines $D_1E_1, D_2E_2, \ldots$ parallel to BC with $E_1, E_2, \ldots$ on $[AC]$.

Then all the segments

$$[AE_1], [E_1E_2], [E_2E_3], \ldots, [E_{m+n-1}C]$$

have the same length, [Theorem 11]
and $E_m = E$ is the point where l cuts $[AC]$. [Axiom of Parallels]
Hence E divides $[AC]$ in the ratio $m : n$. □

Proposition 7. *If two triangles ΔABC and $\Delta A'B'C'$ have*

$$|\angle A| = |\angle A'|, \text{ and } \frac{|A'B'|}{|AB|} = \frac{|A'C'|}{|AC|},$$

then they are similar.

Proof. Suppose $|A'B'| \leq |AB|$. If equal, use SAS. Otherwise, note that then $|A'B'| < |AB|$ and $|A'C'| < |AC|$. Pick B'' on $[AB$ and C'' on $[AC$ with $|A'B'| = |AB''|$ and $|A'C'| = |AC''|$. [Ruler Axiom] Then by SAS, $\Delta A'B'C'$ is congruent to $\Delta AB''C''$.

Draw $[B''D$ parallel to BC [Axiom of Parallels], and let it cut AC at D. Now the last theorem and the hypothesis tell us that D and C'' divide $[AC]$ in the same ratio, and hence $D = C''$.

Thus

$$\begin{aligned} |\angle B| &= |\angle AB''C''| \text{ [Corresponding Angles]} \\ &= |\angle B'|, \end{aligned}$$

and

$$|\angle C| = |\angle AC''B''| = |\angle C'|,$$

so ΔABC is similar to $\Delta A'B'C'$. [Definition of similar]

□

Remark 5. The **Converse to Theorem 12** is true:
Let ΔABC be a triangle. If a line l cuts the sides AB and AC in the same ratio, then it is parallel to BC.

Proof. This is immediate from Proposition 7 and Theorem 5. □

Theorem 13. *If two triangles ΔABC and $\Delta A'B'C'$ are similar, then their sides are proportional, in order:*

$$\frac{|AB|}{|A'B'|} = \frac{|BC|}{|B'C'|} = \frac{|CA|}{|C'A'|}.$$

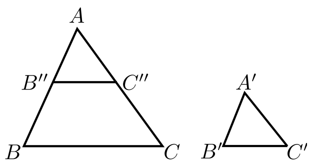

Figure 18.

Proof. We may suppose $|A'B'| \leq |AB|$. Pick B'' on $[AB]$ with $|AB''| = |A'B'|$, and C'' on $[AC]$ with $|AC''| = |A'C'|$. Refer to Figure 18. Then

$\Delta AB''C''$	is congruent to	$\Delta A'B'C'$	[SAS]
$\therefore |\angle AB''C''|$	$=$	$|\angle ABC|$	
$\therefore B''C''$	$\parallel$	BC	[Corresponding Angles]
$\therefore \dfrac{|A'B'|}{|A'C'|}$	$=$	$\dfrac{|AB''|}{|AC''|}$	[Choice of B'', C'']
	$=$	$\dfrac{|AB|}{|AC|}$	[Theorem 12]
$\dfrac{|AC|}{|A'C'|}$	$=$	$\dfrac{|AB|}{|A'B'|}$	[Re-arrange]

Similarly, $\dfrac{|BC|}{|B'C'|} = \dfrac{|AB|}{|A'B'|}$ □

Proposition 8 (Converse)**.** *If*

$$\frac{|AB|}{|A'B'|} = \frac{|BC|}{|B'C'|} = \frac{|CA|}{|C'A'|},$$

then the two triangles ΔABC and $\Delta A'B'C'$ are similar.

Proof. Refer to Figure 18. If $|A'B'| = |AB|$, then by SSS the two triangles are congruent, and therefore similar. Otherwise, assuming $|A'B'| < |AB|$, choose B'' on AB and C'' on AC with $|AB''| = |A'B'|$ and $|AC''| = |A'C'|$. Then by Proposition 7, $\Delta AB''C''$ is similar to ΔABC, so

$$|B''C''| = |AB''| \cdot \frac{|BC|}{|AB|} = |A'B'| \cdot \frac{|BC|}{|AB|} = |B'C'|.$$

Thus by SSS, $\Delta A'B'C'$ is congruent to $\Delta AB''C''$, and hence similar to ΔABC. □

6.9 Pythagoras

Theorem 14 (Pythagoras)**.** *In a right-angle triangle the square of the hypotenuse is the sum of the squares of the other two sides.*

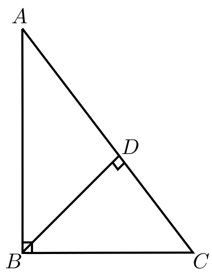

Figure 19.

Proof. Let ΔABC have a right angle at B. Draw the perpendicular BD from the vertex B to the hypotenuse AC (shown in Figure 19).

The right-angle triangles ΔABC and ΔADB have a common angle at A.
$\therefore$ ΔABC is similar to ΔADB.

$$\therefore \frac{|AC|}{|AB|} = \frac{|AB|}{|AD|},$$

so

$$|AB|^2 = |AC| \cdot |AD|.$$

Similarly, ΔABC is similar to ΔBDC.

$$\therefore \frac{|AC|}{|BC|} = \frac{|BC|}{|DC|},$$

so

$$|BC|^2 = |AC| \cdot |DC|.$$

Thus

$$\begin{aligned} |AB|^2 + |BC|^2 &= |AC| \cdot |AD| + |AC| \cdot |DC| \\ &= |AC| (|AD| + |DC|) \\ &= |AC| \cdot |AC| \\ &= |AC|^2. \end{aligned}$$

□

Theorem 15 (Converse to Pythagoras)**.** *If the square of one side of a triangle is the sum of the squares of the other two, then the angle opposite the first side is a right angle.*

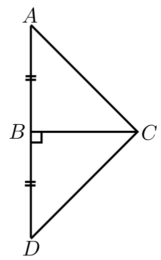

Figure 20.

Proof. (Idea: Construct a second triangle on the other side of [BC], and use Pythagoras and SSS to show it congruent to the original.)
In detail: We wish to show that $|\angle ABC| = 90°$.

Draw $BD \perp BC$ and make $|BD| = |AB|$ (as shown in Figure 20).

Then

$$\begin{aligned}|DC| &= \sqrt{|DC|^2} \\ &= \sqrt{|BD|^2 + |BC|^2} && \text{[Pythagoras]} \\ &= \sqrt{|AB|^2 + |BC|^2} && [\ |AB| = |BD|\ ] \\ &= \sqrt{|AC|^2} && \text{[Hypothesis]} \\ &= |AC|.\end{aligned}$$

$\therefore \Delta ABC$ is congruent to ΔDBC. [SSS]
$\therefore |\angle ABC| = |\angle DBC| = 90°$. □

Proposition 9 (RHS)**.** *If two right angled triangles have hypotenuse and another side equal in length, respectively, then they are congruent.*

Proof. Suppose ΔABC and $\Delta A'B'C'$ are right-angle triangles, with the right angles at B and B', and have hypotenuses of the same length, $|AC| = |A'C'|$, and also have $|AB| = |A'B'|$. Then by using Pythagoras' Theorem, we obtain $|BC| = |B'C'|$, so by SSS, the triangles are congruent. □

Proposition 10. *Each point on the perpendicular bisector of a segment* $[AB]$ *is equidistant from the ends.*

Proposition 11. *The perpendiculars from a point on an angle bisector to the arms of the angle have equal length.*

6.10 Area

Definition 37. If one side of a triangle is chosen as the base, then the opposite vertex is the **apex** corresponding to that base. The corresponding **height** is the length of the perpendicular from the apex to the base. This perpendicular segment is called an **altitude** of the triangle.

Theorem 16. *For a triangle, base times height does not depend on the choice of base.*

Proof. Let AD and BE be altitudes (shown in Figure 21). Then ΔBCE and ΔACD are right-angled triangles that share the angle C, hence they are similar. Thus

$$\frac{|AD|}{|BE|} = \frac{|AC|}{|BC|}.$$

Re-arrange to yield the result. □

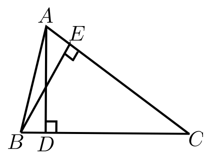

Figure 21.

Definition 38. The **area** of a triangle is half the base by the height.

Notation 5. We denote the area by "area of ΔABC"[19].

Proposition 12. *Congruent triangles have equal areas.*

Remark 6. This is another example of a proposition whose converse is false. It may happen that two triangles have equal area, but are not congruent.

Proposition 13. *If a triangle ΔABC is cut into two by a line AD from A to a point D on the segment $[BC]$, then the areas add up properly:*

$$\text{area of } \Delta ABC = \text{area of } \Delta ABD + \text{area of } \Delta ADC.$$

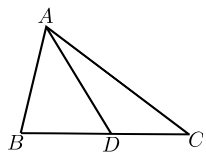

Figure 22.

Proof. See Figure 22. All three triangles have the same height, say h, so it comes down to

$$\frac{|BC| \times h}{2} = \frac{|BD| \times h}{2} + \frac{|DC| \times h}{2},$$

which is obvious, since

$$|BC| = |BD| + |DC|.$$

□

[19] $|\Delta ABC|$ will also be accepted.

If a figure can be cut up into nonoverlapping triangles (i.e. triangles that either don't meet, or meet only along an edge), then its area is taken to be the sum of the area of the triangles[20].

If figures of equal areas are added to (or subtracted from) figures of equal areas, then the resulting figures also have equal areas[21].

Proposition 14. *The area of a rectangle having sides of length a and b is ab.*

Proof. Cut it into two triangles by a diagonal. Each has area $\frac{1}{2}ab$. □

Theorem 17. *A diagonal of a parallelogram bisects the area.*

Proof. A diagonal cuts the parallelogram into two congruent triangles, by Corollary 1. □

Definition 39. Let the side AB of a parallelogram $ABCD$ be chosen as a base (Figure 23). Then the **height** of the parallelogram **corresponding to that base** is the height of the triangle ΔABC.

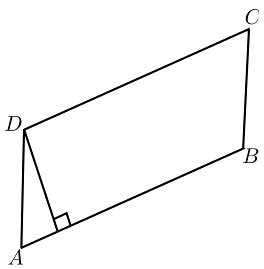

Figure 23.

Proposition 15. *This height is the same as the height of the triangle ΔABD, and as the length of the perpendicular segment from D onto AB.*

[20] If students ask, this does not lead to any ambiguity. In the case of a convex quadrilateral, $ABCD$, one can show that

$$\text{area of } \Delta ABC + \text{area of } \Delta CDA = \text{area of } \Delta ABD + \text{area of } \Delta BCD.$$

In the general case, one proves the result by showing that there is a common refinement of any two given triangulations.

[21] Follows from the previous footnote.

Theorem 18. *The area of a parallelogram is the base by the height.*

Proof. Let the parallelogram be $ABCD$. The diagonal BD divides it into two triangles, ΔABD and ΔCDB. These have equal area, [Theorem 17] and the first triangle shares a base and the corresponding height with the parallelogram. So the areas of the two triangles add to $2 \times \frac{1}{2} \times \text{base} \times \text{height}$, which gives the result. □

6.11 Circles

Definition 40. A **circle** is the set of points at a given distance (its **radius**) from a fixed point (its **centre**). Each line segment joining the centre to a point of the circle is also called **a** radius. The plural of radius is radii. A **chord** is the segment joining two points of the circle. A **diameter** is a chord through the centre. All diameters have length twice the radius. This number is also called **the** diameter of the circle.

Two points A, B on a circle cut it into two pieces, called **arcs**. You can specify an arc uniquely by giving its endpoints A and B, and one other point C that lies on it. A **sector** of a circle is the piece of the plane enclosed by an arc and the two radii to its endpoints.

The length of the whole circle is called its **circumference**. For every circle, the circumference divided by the diameter is the same. This ratio is called π.

A **semicircle** is an arc of a circle whose ends are the ends of a diameter.

Each circle divides the plane into two pieces, the inside and the outside. The piece inside is called a **disc**.

If B and C are the ends of an arc of a circle, and A is another point, not on the arc, then we say that the angle $\angle BAC$ is the angle at A **standing on the arc**. We also say that it **stands on the chord** $[BC]$.

Theorem 19. *The angle at the centre of a circle standing on a given arc is twice the angle at any point of the circle standing on the same arc.*

Proof. There are several cases for the diagram. It will be sufficient for students to examine one of these. The idea, in all cases, is to draw the line through the centre and the point on the circumference, and use the Isosceles Triangle Theorem, and then the Protractor Axiom (to add or subtract angles, as the case may be).

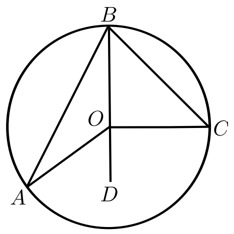

Figure 24.

In detail, for the given figure, Figure 24, we wish to show that $|\angle AOC| = 2|\angle ABC|$.

Join B to O and continue the line to D. Then

$$\begin{array}{rcll} |OA| &=& |OB|. & \text{[Definition of circle]} \\ \therefore |\angle BAO| &=& |\angle ABO|. & \text{[Isosceles triangle]} \\ \therefore |\angle AOD| &=& |\angle BAO| + |\angle ABO| & \text{[Exterior Angle]} \\ &=& 2 \cdot |\angle ABO|. & \end{array}$$

Similarly,

$$|\angle COD| = 2 \cdot |\angle CBO|.$$

Thus

$$\begin{array}{rcl} |\angle AOC| &=& |\angle AOD| + |\angle COD| \\ &=& 2 \cdot |\angle ABO| + 2 \cdot |\angle CBO| \\ &=& 2 \cdot |\angle ABC|. \end{array}$$

□

Corollary 2. *All angles at points of the circle, standing on the same arc, are equal. In symbols, if A, A', B and C lie on a circle, and both A and A' are on the same side of the line BC, then $\angle BAC = \angle BA'C$.*

Proof. Each is half the angle subtended at the centre. □

Remark 7. The converse is true, but one has to careful about sides of BC:

Converse to Corollary 2: *If points A and A' lie on the same side of the line BC, and if $|\angle BAC| = |\angle BA'C|$, then the four points A, A', B and C lie on a circle.*

Proof. Consider the circle s through A, B and C. If A' lies outside the circle, then take A'' to be the point where the segment $[A'B]$ meets s. We then have

$$|\angle BA'C| = |\angle BAC| = |\angle BA''C|,$$

by Corollary 2. This contradicts Theorem 6.

A similar contradiction arises if A' lies inside the circle. So it lies on the circle. □

Corollary 3. *Each angle in a semicircle is a right angle. In symbols, if BC is a diameter of a circle, and A is any other point of the circle, then $\angle BAC = 90°$.*

Proof. The angle at the centre is a straight angle, measuring 180°, and half of that is 90°. □

Corollary 4. *If the angle standing on a chord $[BC]$ at some point of the circle is a right angle, then $[BC]$ is a diameter.*

Proof. The angle at the centre is 180°, so is straight, and so the line BC passes through the centre. □

Definition 41. A **cyclic** quadrilateral is one whose vertices lie on some circle.

Corollary 5. *If $ABCD$ is a cyclic quadrilateral, then opposite angles sum to 180°.*

Proof. The two angles at the centre standing on the same arcs add to 360°, so the two halves add to 180°. □

Remark 8. The converse also holds: *If $ABCD$ is a convex quadrilateral, and opposite angles sum to 180°, then it is cyclic.*

Proof. This follows directly from Corollary 5 and the converse to Corollary 2. □

It is possible to approximate a disc by larger and smaller equilateral polygons, whose area is as close as you like to πr^2, where r is its radius. For this reason, we say that the area of the disc is πr^2.

Proposition 16. *If l is a line and s a circle, then l meets s in zero, one, or two points.*

Proof. We classify by comparing the length p of the perpendicular from the centre to the line, and the radius r of the circle. If $p > r$, there are no points. If $p = r$, there is exactly one, and if $p < r$ there are two. □

Definition 42. The line l is called a **tangent** to the circle s when $l \cap s$ has exactly one point. The point is called the **point of contact** of the tangent.

Theorem 20.
(1) Each tangent is perpendicular to the radius that goes to the point of contact.
(2) If P lies on the circle s, and a line l through P is perpendicular to the radius to P, then l is tangent to s.

Proof. (1) This proof is a proof by contradiction.

Suppose the point of contact is P and the tangent l is not perpendicular to OP.

Let the perpendicular to the tangent from the centre O meet it at Q. Pick R on PQ, on the other side of Q from P, with $|QR| = |PQ|$ (as in Figure 25).

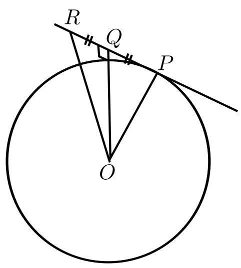

Figure 25.

Then ΔOQR is congruent to ΔOQP. [SAS]

$$\therefore |OR| = |OP|,$$

so R is a second point where l meets the circle. This contradicts the given fact that l is a tangent.

Thus l must be perpendicular to OP, as required.

(2) (Idea: Use Pythagoras. This shows directly that each other point on l is further from O than P, and hence is not on the circle.)

In detail: Let Q be any point on l, other than P. See Figure 26. Then

$$\begin{aligned} |OQ|^2 &= |OP|^2 + |PQ|^2 && \text{[Pythagoras]} \\ &> |OP|^2. \\ \therefore |OQ| &> |OP|. \end{aligned}$$

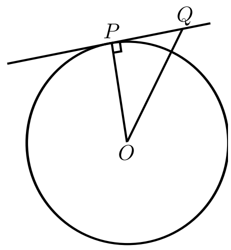

Figure 26.

$\therefore Q$ is not on the circle. [Definition of circle]
$\therefore P$ is the only point of l on the circle.
$\therefore l$ is a tangent. [Definition of tangent]
□

Corollary 6. *If two circles share a common tangent line at one point, then the two centres and that point are collinear.*

Proof. By part (1) of the theorem, both centres lie on the line passing through the point and perpendicular to the common tangent. □

The circles described in Corollary 6 are shown in Figure 27.

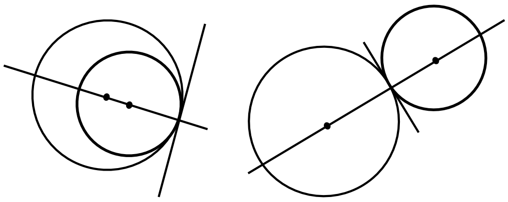

Figure 27.

Remark 9. Any two distinct circles will intersect in 0, 1, or 2 points.

If they have two points in common, then the common chord joining those two points is perpendicular to the line joining the centres.

If they have just one point of intersection, then they are said to be *touching* and this point is referred to as their *point of contact*. The centres and the point of contact are collinear, and the circles have a common tangent at that point.

Theorem 21.
(1) The perpendicular from the centre to a chord bisects the chord.
(2) The perpendicular bisector of a chord passes through the centre.

Proof. (1) (Idea: Two right-angled triangles with two pairs of sides equal.) See Figure 28.

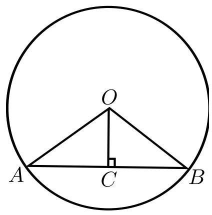

Figure 28.

In detail:

$$\begin{aligned} |OA| &= |OB| && \text{[Definition of circle]} \\ |OC| &= |OC| && \\ |AC| &= \sqrt{|OA|^2 - |OC|^2} && \text{[Pythagoras]} \\ &= \sqrt{|OB|^2 - |OC|^2} && \\ &= |CB|. && \text{[Pythagoras]} \end{aligned}$$

$\therefore \Delta OAC$ is congruent to ΔOBC. [SSS]
$\therefore |AC| = |CB|$.

(2) This uses the Ruler Axiom, which has the consequence that a segment has exactly one midpoint.
Let C be the foot of the perpendicular from O on AB.
By Part (1), $|AC| = |CB|$, so C is the midpoint of $[AB]$.
Thus CO is the perpendicular bisector of AB.
Hence the perpendicular bisector of AB passes through O. □

6.12 Special Triangle Points

Proposition 17. *If a circle passes through three non-collinear points A, B, and C, then its centre lies on the perpendicular bisector of each side of the triangle ΔABC.*

Definition 43. The **circumcircle** of a triangle ΔABC is the circle that passes through its vertices (see Figure 29). Its centre is the **circumcentre** of the triangle, and its radius is the **circumradius**.

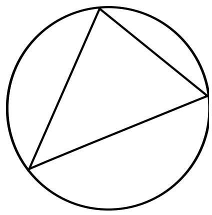

Figure 29.

Proposition 18. *If a circle lies inside the triangle ΔABC and is tangent to each of its sides, then its centre lies on the bisector of each of the angles $\angle A$, $\angle B$, and $\angle C$.*

Definition 44. The **incircle** of a triangle is the circle that lies inside the triangle and is tangent to each side (see Figure 30). Its centre is the **incentre**, and its radius is the **inradius**.

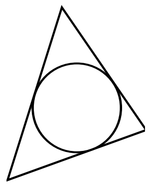

Figure 30.

Proposition 19. *The lines joining the vertices of a triangle to the centre of the opposite sides meet in one point.*

Definition 45. A line joining a vertex of a triangle to the midpoint of the opposite side is called a **median** of the triangle. The point where the three medians meet is called the **centroid**.

Proposition 20. *The perpendiculars from the vertices of a triangle to the opposite sides meet in one point.*

Definition 46. The point where the perpendiculars from the vertices to the opposite sides meet is called the **orthocentre** (see Figure 31).

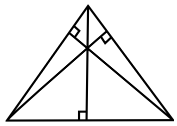

Figure 31.

7 Constructions to Study

The instruments that may be used are:

straight-edge: This may be used (together with a pencil) to draw a straight line passing through two marked points.

compass: This instrument allows you to draw a circle with a given centre, passing through a given point. It also allows you to take a given segment $[AB]$, and draw a circle centred at a given point C having radius $|AB|$.

ruler: This is a straight-edge marked with numbers. It allows you measure the length of segments, and to mark a point B on a given ray with vertex A, such that the length $|AB|$ is a given positive number. It can also be employed by sliding it along a set square, or by other methods of sliding, while keeping one or two points on one or two curves.

protractor: This allows you to measure angles, and mark points C such that the angle $\angle BAC$ made with a given ray $[AB$ has a given number of degrees. It can also be employed by sliding it along a line until some line on the protractor lies over a given point.

set-squares: You may use these to draw right angles, and angles of $30°$, $60°$, and $45°$. It can also be used by sliding it along a ruler until some coincidence occurs.

The prescribed constructions are:

1. Bisector of a given angle, using only compass and straight edge.
2. Perpendicular bisector of a segment, using only compass and straight edge.
3. Line perpendicular to a given line l, passing through a given point not on l.

4. Line perpendicular to a given line l, passing through a given point on l.
5. Line parallel to given line, through given point.
6. Division of a segment into 2, 3 equal segments, without measuring it.
7. Division of a segment into any number of equal segments, without measuring it.
8. Line segment of given length on a given ray.
9. Angle of given number of degrees with a given ray as one arm.
10. Triangle, given lengths of three sides.
11. Triangle, given SAS data.
12. Triangle, given ASA data.
13. Right-angled triangle, given the length of the hypotenuse and one other side.
14. Right-angled triangle, given one side and one of the acute angles (several cases).
15. Rectangle, given side lengths.
16. Circumcentre and circumcircle of a given triangle, using only straight-edge and compass.
17. Incentre and incircle of a given triangle, using only straight-edge and compass.
18. Angle of 60°, without using a protractor or set square.
19. Tangent to a given circle at a given point on it.
20. Parallelogram, given the length of the sides and the measure of the angles.
21. Centroid of a triangle.
22. Orthocentre of a triangle.

8 Teaching Approaches

8.1 Practical Work

Practical exercises and experiments should be undertaken before the study of theory. These should include:

1. Lessons along the lines suggested in the Guidelines for Teachers [2]. We refer especially to Section 4.6 (7 lessons on Applied Arithmetic and Measure), Section 4.9 (14 lessons on Geometry), and Section 4.10 (4 lessons on Trigonometry).

2. Ideas from Technical Drawing.

3. Material in [3].

8.2 From Discovery to Proof

It is intended that all of the geometrical results on the course would first be encountered by students through investigation and discovery. As a result of various activities undertaken, students should come to appreciate that certain features of certain shapes or diagrams appear to be independent of the particular examples chosen. These apparently constant features therefore seem to be general results that we have reason to believe might always be true. At this stage in the work, we ask students to accept them as true for the purpose of applying them to various contextualised and abstract problems, but we also agree to come back later to revisit this question of their truth. Nonetheless, even at this stage, students should be asked to consider whether investigating a number of examples in this way is sufficient to be convinced that a particular result always holds, or whether a more convincing argument is required. Is a person who refuses to believe that the asserted result will always be true being unreasonable? An investigation of a statement that appears at first to be always true, but in fact is not, may be helpful, (e.g. the assertion that $n^2 + n + 41$ is prime for all $n \in \mathbb{N}$). Reference might be made to other examples of conjectures that were historically believed to be true until counterexamples were found.

Informally, the ideas involved in a mathematical proof can be developed even at this investigative stage. When students engage in activities that lead to closely related results, they may readily come to appreciate the manner in which these results are connected to each other. That is, they may see for themselves or be led to see that the result they discovered today is an inevitable logical consequence of the one they discovered yesterday. Also, it should be noted that working on problems or "cuts" involves logical deduction from general results.

Later, students at the relevant levels need to proceed beyond accepting a result on the basis of examples towards the idea of a more convincing logical argument. Informal justifications, such as a dissection-based proof of Pythagoras' theorem, have a role to play here. Such justifications develop an argument more strongly than a set of examples. It is worth discussing what the word "prove" means in various contexts, such as in a criminal trial, or in a civil court, or in everyday language. What mathematicians regard as a "proof" is quite different from these other contexts. The logic involved in the various steps must be unassailable. One might present one or more of the readily available dissection-based "proofs" of fallacies and then probe a dissection-based proof of Pythagoras' theorem to see what possible gaps might need to be bridged.

As these concepts of argument and proof are developed, students should be led to appreciate the need to formalise our idea of a mathematical proof to lay out the ground rules that we can all agree on. Since a formal proof only allows us to progress logically from existing results to new ones, the need for axioms is readily identified, and the students can be introduced to formal proofs.

9 JCOL Content

9.1 Concepts

Set, subset, plane, point, line, ray, angle, real number, length,

degree, triangle, right-angle, congruent triangles, similar triangles, parallel lines, parallelogram, area,

segment, collinear points, distance, midpoint of a segment,

reflex angle, ordinary angle, straight angle, null angle, full angle, supplementary angles, vertically-opposite angles, acute angle, obtuse angle,

angle bisector, perpendicular lines, perpendicular bisector of a segment, ratio,

isosceles triangle, equilateral triangle, scalene triangle, right-angled triangle, exterior angles of a triangle, interior opposite angles, hypotenuse, alternate angles, corresponding angles,

polygon, quadrilateral, convex quadrilateral, rectangle, square, rhombus, base and corresponding apex and height of triangle or parallelogram, transversal line,

circle, centre of a circle, radius, diameter, chord, arc, sector, circumference of a circle, disc, area of a disc, circumcircle, tangent to a circle, point of contact of a tangent,

vertex, vertices (of angle, triangle, polygon), endpoints of segment, arms of an angle, equal segments, equal angles, adjacent sides, angles, or vertices of triangles or quadrilaterals, the side opposite an angle of a triangle, opposite sides or angles of a quadrilateral.

9.2 Constructions

Students will study constructions 1, 2, 4, 5, 6, 8, 9, 10, 11, 12, 13, 14, 15.

9.3 Axioms and Proofs

Students should be exposed to situations where they are required to use the terms theorem proof, axiom, corollary, converse, and implies. The students should be exposed to some formal proofs. They will not be examined on these. They will see Axioms 1,2,3,4,5, and study the proofs of Theorems 1, 2, 3, 4, 5, 6, 9, 10, 14, 15; and direct proofs of Corollaries 3, 4. They will study the statement and use of Theorem 13, but need not study its formal proof.

10 Additional Content for JCHL

10.1 Concepts

Concurrent lines.

10.2 Constructions

Constructions 3 and 7.

10.3 Logic, Axioms and Theorems

Students should be exposed to situations where they are required to use and explain the terms Theorem, proof, axiom, corollary, converse, implies.

They will study Axioms 1, 2, 3, 4, 5. They will study the proofs of Theorems 13, 19, Corollaries 1, 2, 3, 4, 5, and their converses.

They will make use of Theorems 11 and 12, but need not study their formal proofs. The formal material on area will not be studied at this level. Students will deal with area only as part of the material on arithmetic and mensuration. They will study geometrical problems.

11 Syllabus for LCFL

Students are expected to build on their mathematical experiences to date.

11.1 Constructions

Students revisit constructions 4, 5, 10, 13, 15, and learn how to apply these in real-life contexts.

12 Syllabus for LCOL

12.1 Constructions

A knowledge of the constructions prescribed for JC-OL will be assumed, and may be examined. In addition, students will study constructions 16–21.

12.2 Theorems and Proofs

Students will be expected to understand the meaning of the following terms related to logic and deductive reasoning: **Theorem, proof, axiom, corollary, converse, implies.**

A knowledge of the Axioms, concepts, Theorems and Corollaries prescribed for JC-OL will be assumed.

Students will study proofs of Theorems 7, 8, 11, 12, 13, 16, 17, 18, 20, 21, and Corollary 6.

No proofs are examinable. Students will be examined using problems that can be attacked using the theory.

13 Syllabus for LCHL

13.1 Constructions

A knowledge of the constructions prescribed for JC-HL will be assumed, and may be examined. In addition, students will study the constructions prescribed for LC-OL, and construction 22.

13.2 Theorems and Proofs

Students will be expected to understand the meaning of the following terms related to logic and deductive reasoning: **Theorem, proof, axiom, corollary, converse, implies, is equivalent to, if and only if, proof by contradiction.**

A knowledge of the Axioms, concepts, Theorems and Corollaries prescribed for JC-HL will be assumed.

Students will study all the theorems and corollaries prescribed for LC-OL, but will not, in general, be asked to reproduce their proofs in examination.

However, they may be asked to give proofs of the Theorems 11, 12, 13, concerning ratios, which lay the proper foundation for the proof of Pythagoras studied at JC, and for trigonometry.

They will be asked to solve geometrical problems (so-called "cuts") and write reasoned accounts of the solutions. These problems will be such that they can be attacked using the given theory. The study of the propositions may be a useful way to prepare for such examination questions.

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[2] Junior Cycle Course Committee, NCCA. *Mathematics: Junior Certificate Guidelines for Teachers.* Stationary Office, Dublin. 2002. ISBN 0-7557-1193-9.

[3] Fiacre O'Cairbre, John McKeon, and Richard O. Watson. A Resource for Transition Year Mathematics Teachers. DES. Dublin. 2006.

[4] Anthony G. O'Farrell. *School Geometry.* IMTA Newsletter 109 (2009) 21-28.

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An Chomhairle Náisiúnta Curaclaim agus Measúnachta
National Council for Curriculum and Assessment

An Roinn Oideachais agus Scileanna
Department of Education and Skills