Oide

Tacú leis an bhFoghlaim Ghairmiúil i measc Ceannairí Scoile agus Múinteoirí

Supporting the Professional Learning of School Leaders and Teachers

Applied Mathematics

Professional Learning Booklet

2023-2024

Tacú leis an bhFoghlaim Ghairmiúil i measc Ceannairí Scoile agus Múinteoirí

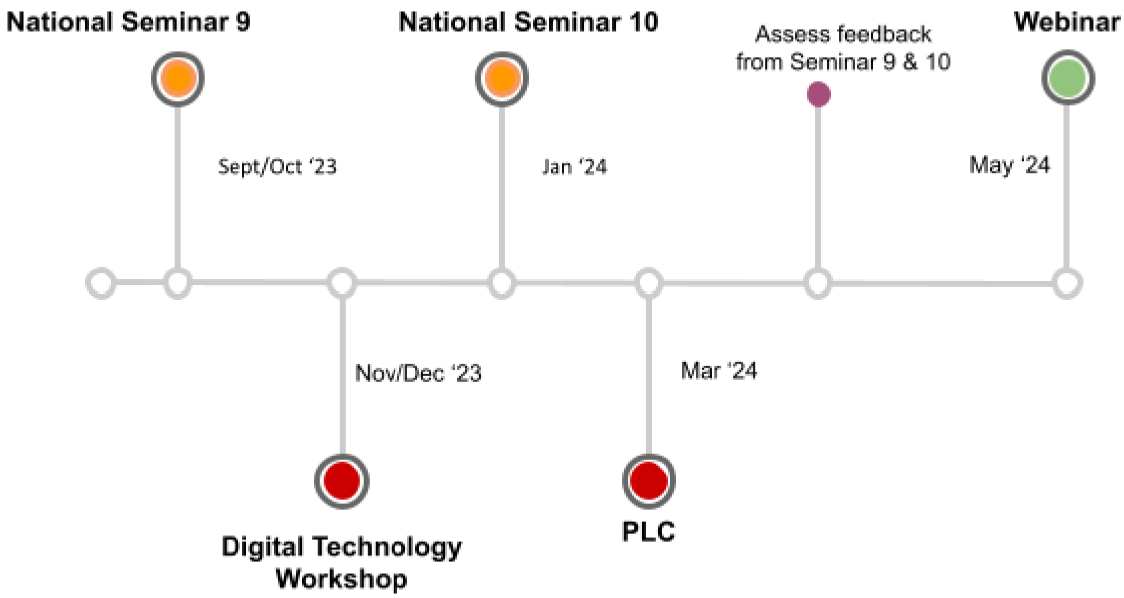
Supporting the Professional Learning of School Leaders and Teachers

Introduction

Schedule

09:30 - 11:00	Reviewing the journey so far and supporting students with the modelling project
11:00 - 11:15	Tea and Coffee
11:15 - 13:00	Modelling with Multi-Stage Dynamic Programming
13:00 - 14:00	Lunch
14:00 - 15:30	Exploring Difference Equations

Overview Of Professional Development



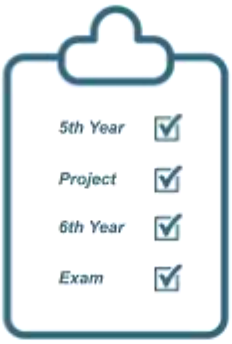
Key Messages

1. Core to the specification is a non-linear approach empowered by the use of rich pedagogy which promotes the making of connections between various Applied Mathematics learning outcomes.

2. Strand 1 of the specification is a unifying strand and emphasises the importance of utilising modelling across all learning outcomes.

3. Applied Mathematics is rooted in authentic problems as a context for learning about the application of Mathematics to design solutions for real-world problems and to develop problem solving skills applicable to a variety of disciplines.

Session 1



Discussion - Taking Stock

Now that the first two year cycle of teaching the specification has been completed,

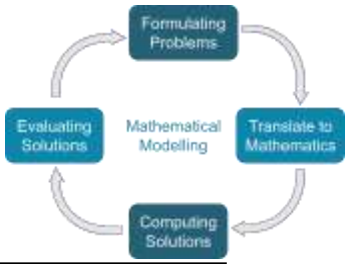
What have your main takeaways been?

What has been your biggest learning as a teacher?

Preparing For The Modelling Project

Having supported students in completing the first modelling project in 23/24,

What three nuggets of wisdom would you give a teacher who is engaging with it for the first time this year?



Supporting Students With The Project

How best can teachers support students,

before …

during …

after …

the modelling project?

Modelling Problem

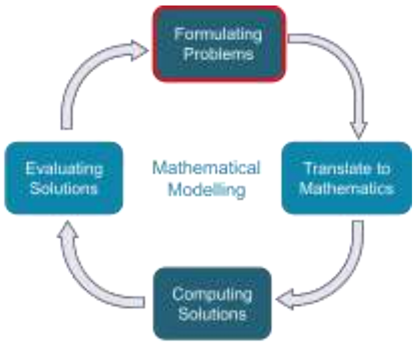
Complete a mathematical modelling problem based on the following context:

The 2024 European football Championship takes place at multiple venues across Germany in June/July. A key feature of a team's preparation for this is planning the logistics of travel, accommodation, purchasing and allocating stock for the team and scheduling a team's itinerary.

Select one or more aspects of logistical planning and model the problem(s) you have selected using The Modelling Cycle.

Formulating The Problem

What is your problem statement and what research must you do?
What variables (factors) are relevant to the problem?
Can you simplify the problem into smaller manageable parts?
Consider if there are limitations to your model due to your chosen assumptions?
Can you predict what the output of your model will achieve?



What problem statement could students initially choose to investigate?

What research and assumptions would be required for students?

Advice From Other Courseworks

Candidates need to make a greater link(s) between their particular investigation and their background research conducted.

Candidates need more experience with research methods and the compiling of research which will aid them in the completion of their coursework. They should avoid just "copying and pasting" external material

The analysis needs to be in-depth. Reporting research findings requires more than a summary or interpretation. Critical evaluation of data is required so that candidates can then form an argument or a judgment.

Candidates should ensure that their research is based on a clear and focused line of inquiry which provides opportunities for comprehensive analysis, development of arguments and judgments, evaluation and conclusions.

Candidates should ensure that they critically reflect on new knowledge or understanding gained, how their thinking, behaviour or opinions have changed or developed since the beginning of the process, and the importance of this.

Plan A Suitable Project Timeline

How will you allocate your class time from when the project is released to when it is submitted?

In groups, discuss an appropriate timeline for students' engagement with the project and how teachers will support them during this timeframe.

Reflection

What were your key takeaways from this session?

How can you implement ideas from this session into your teaching?

What are the next steps for enhancing students’ modelling skills in your classroom?

Session 2

Dynamic Programming with Multi-Stage Authentic Problems

Strand 2 Support

Seminar 1: **Introduction to Networks and Graph Theory, Algorithms and their applications**

Seminar 2: **Development of Dijkstra’s Algorithm through Modelling**

Seminar 4: **Project Scheduling**

Seminar 5: **Bellman’s Principle of Optimality and Dynamic Programming**

Seminar 8: **Exploring Project Scheduling with Project Scheduling Diagrams**

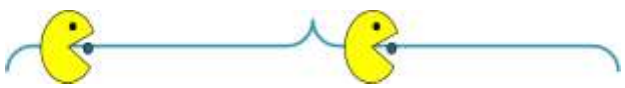
All slides and relevant resources available on:

https://pdst.ie/post-primary/sc/appliedmaths/cpd-resources

Strand 2 Algorithms

Minimum Spanning Tree

Optimization



Prim's

Starts from a single vertex and adds edges one at a time

Generally faster for dense graphs

Kruskal's

Sorts edges by weight and adds them to the tree if they don't create a cycle

Works well with sparse graphs, does not require a starting vertex

Dijkstra's

Finds the shortest path between a source vertex and all other vertices

Breaks down with negative edge weights

Dynamic Programming

Breaks a problem down into smaller sub-problems. Solutions to sub-problems stored and then the solution to overall problem constructed from the solutions to the sub-problems.

Dynamic Programming

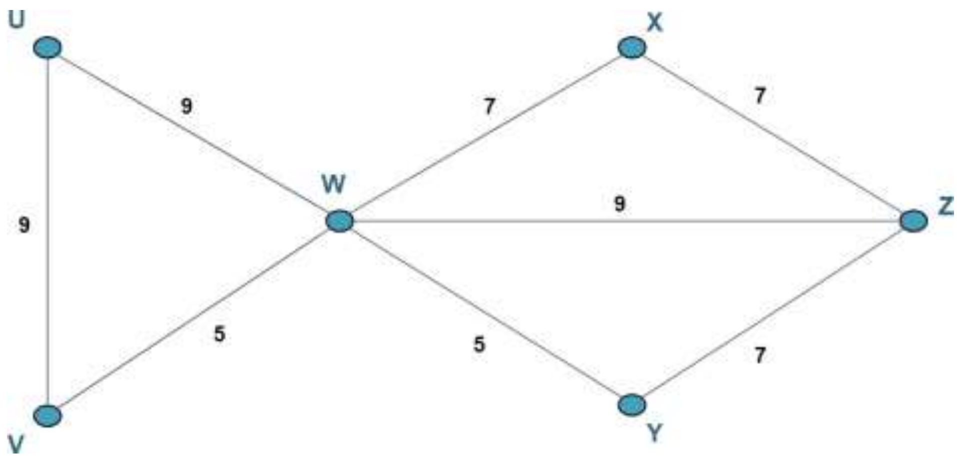
- Dynamic Programming is not greedy
- Uses backward recursion, it takes an overall view of a problem.
- Can handle maximum and minimum problems easily and negative edge weights.
- Easily applicable to problems given in the form of a table.

Main disadvantages: requires a staged network and as it stores sub-problems, the time cost and space required to implement are higher.

Strand 2 Algorithms

Reviewing **Prim's** and **Kruskal's**

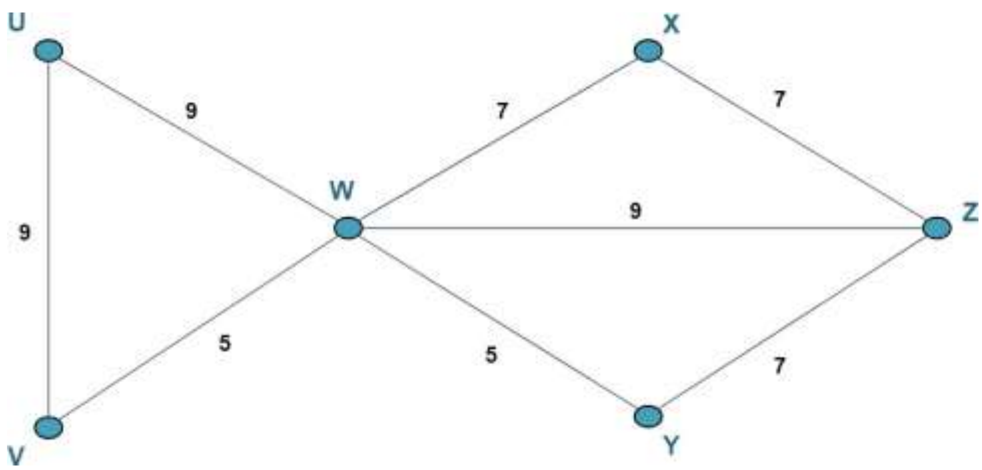
Find a minimum spanning tree for the below network using Prim's and then Kruskal's Algorithm. There are 4 possible solutions.



Strand 2 Algorithms

Reviewing **Dijkstra's** Algorithm

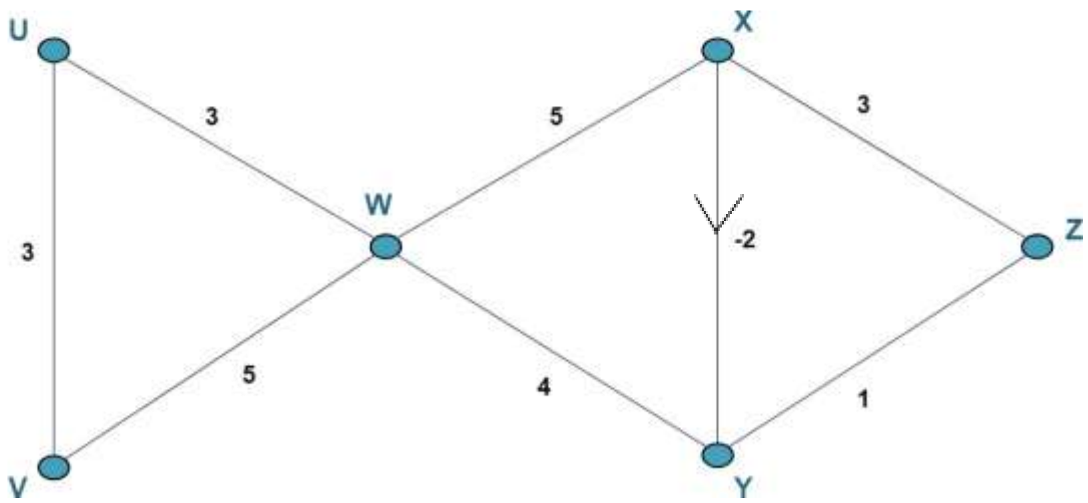
Apply Dijkstra's algorithm to find the shortest path from U to Z.



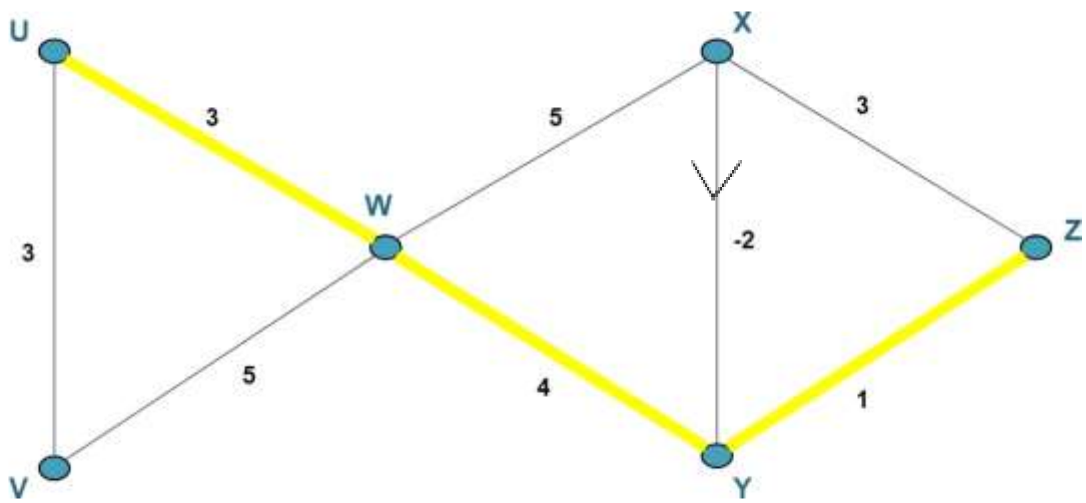
Strand 2 Algorithms

Reviewing **Dijkstra's** Algorithm

Apply Dijkstra's to find the shortest path from U to Z in this network. Does it yield a correct solution?



Applying Dijkstra's gives a shortest path of UWYZ (shown below) with a total weight of 8. Is this correct, Is there a shorter path?



Strand 2 Algorithms

Applying **Dynamic Programming**

Dynamic Programming is based on Bellman's Principle of Optimality

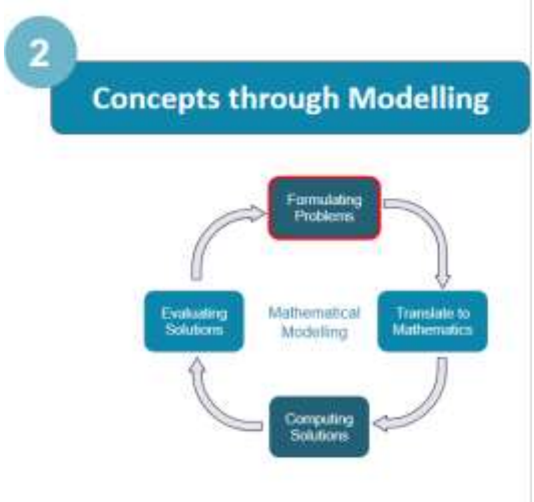
Any part of the shortest/longest path between the source and sink nodes is itself a shortest/longest path

Or: 'any part of the optimal path is itself optimal'

Interpreting a Real-World Problem

In many real-world settings the management of stock is an important consideration.

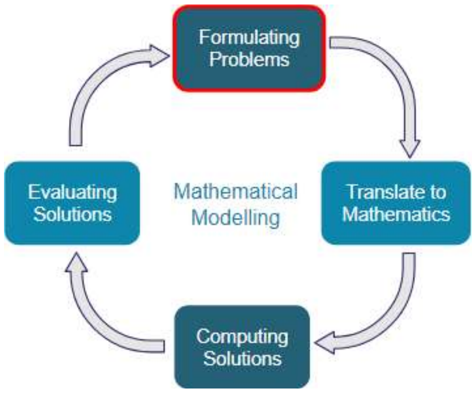
Choose a real-world problem related to the distribution or management of stock and model the problem(s) you have selected using *The Modelling Cycle*.



Interpreting a Real-World Problem

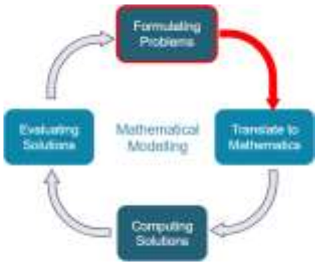
Formulating Problems

Problem Statement: Joystick Junction has the last remaining stock of a new games console. What is the best route to take to get to Joystick Junction on the other side of the city?

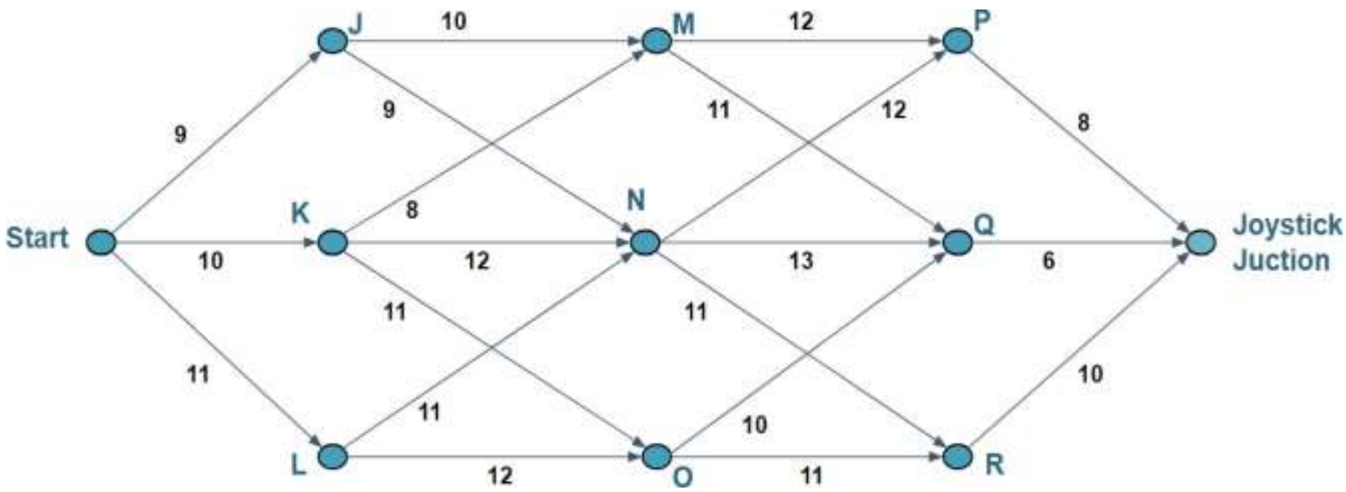


The Modelling Cycle

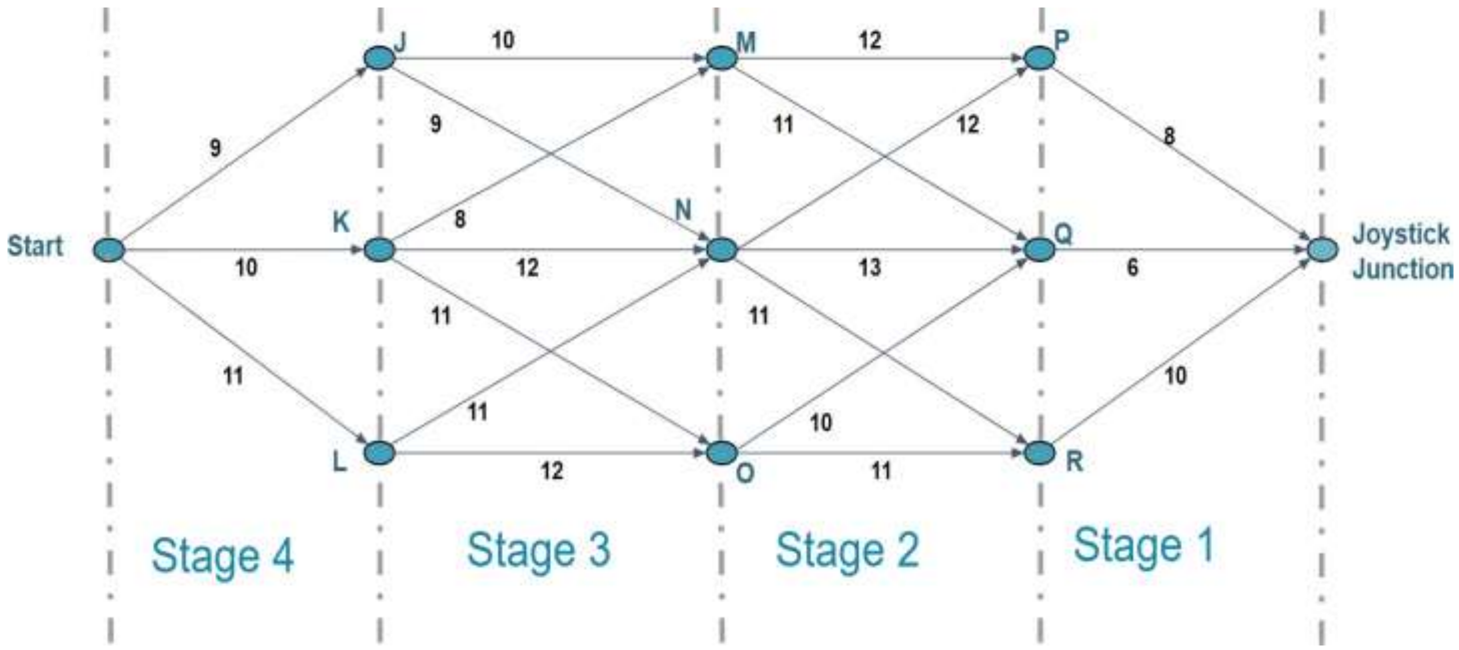
Translating to Mathematics



The city can be represented with a simplified network shown below and Bellman's principle can be applied directly to the network.



This shortest route problem can also be solved using an analogous table method. The first step is to identify the stages working backwards from the end point as shown below.



The Modelling Cycle

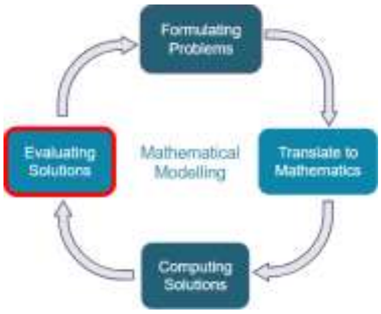
Computing Solutions

Use a table method to find the shortest route to Joystick Junction

Stage	State	Action	Value
1	P	JJ	0+8 = 8*
	Q	JJ	0+6 = 6*
	R	JJ	0+10 = 10*
2	M	P	12 + 8 = 20
		Q	11+6 = 17*
	N	P	
		Q	
		R	
	O	Q	
		R	
3	J	M	
		N	
	K	M	
		N	
		O	
	L	N	
		O	
4	Start	J	
		K	
		L	

The Modelling Cycle

Evaluating Solutions



Interpret your mathematical solution(s) in the context of the problem you are modelling.

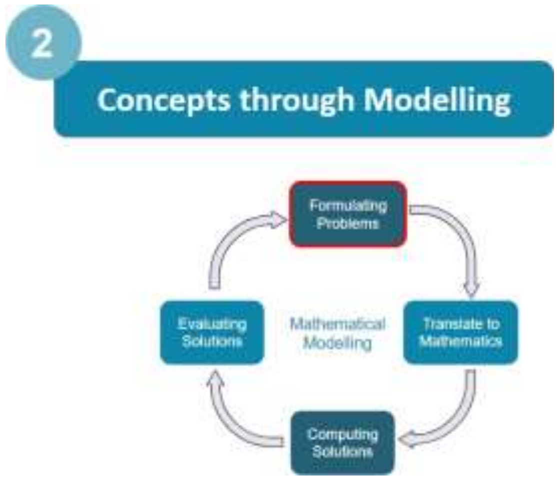
How accurate and reliable is your solution based on your earlier assumptions?

How can you refine your assumptions to improve your solution and how will this change your solution?

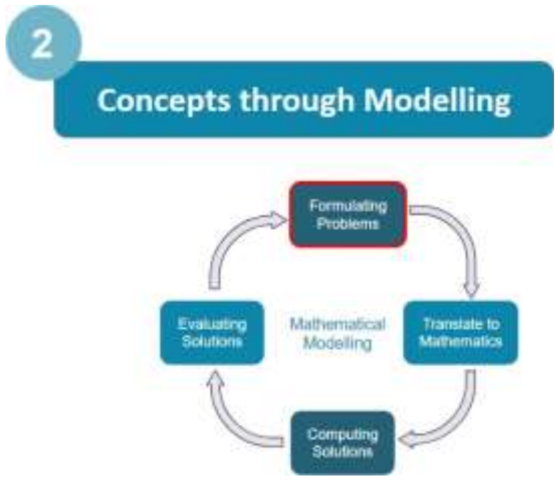
Interpreting a Real-World Problem
Formulating Problems

In many real-world settings the management of stock is an important consideration.

Choose a real-world problem related to the distribution or management of stock and model the problem(s) you have selected using *The Modelling Cycle*.



Problem Statement: How should I allocate stock across a number of retailers in order to maximise profit?

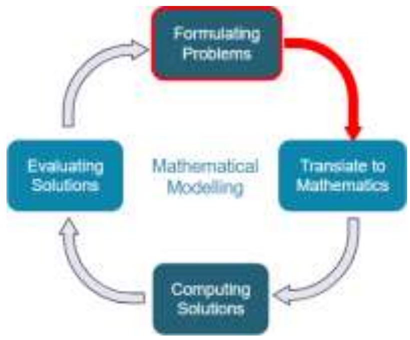


The Modelling Cycle

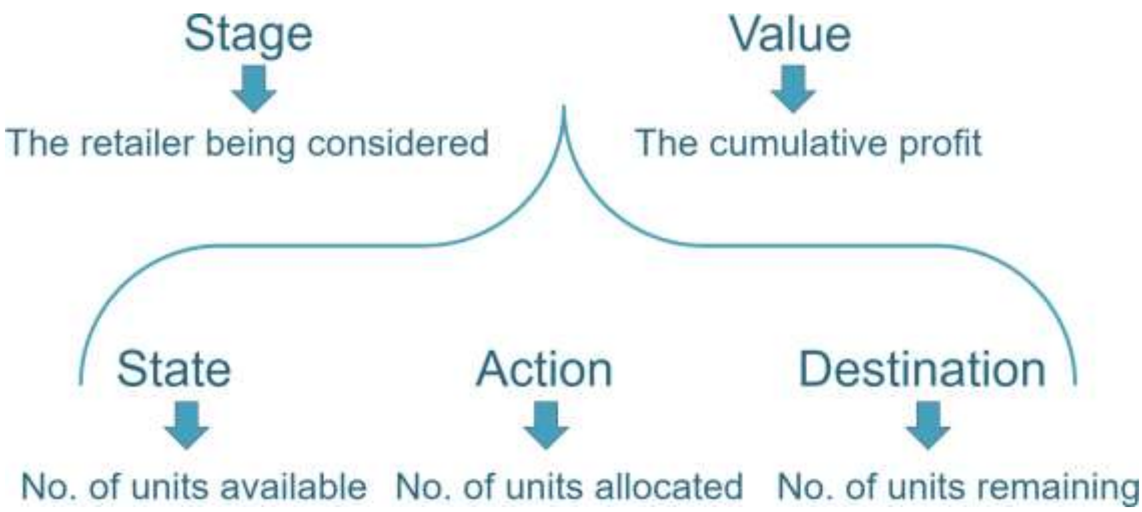
Formulating Problems

A games manufacturer needs to distribute 500 games consoles every month and can allocate these in multiples of 100 to three different retailers. The distributor fee/profit, in €100s, for the number of units allocated to each retailer is shown in the table.

	Number of consoles allocated (x100)				
Retailers	1	2	3	4	5
Joystick Junction	11	25	30	32	33
Button Bashers	15	16	17	18	19
Gamers Grotto	7	14	21	28	35



The manufacturer wants to know how many consoles should be allocated to each retailer to maximise their monthly income.



Use the table to figure out the best way to allocate the 500 consoles in order to maximise the distributor fees/profit.

Stage	State (Units Available)	Action (Units Allocated)	Destination (Units Remaining)	Value (Cumulative Profit)
Gamers Grotto	0	0	0	0*
	1	1	0	7*
	2	2	0	14*
	3	3	0	21*
	4	4	0	28*
	5	5	0	35*
Button Bashers	0	0	0	0 + 0 = 0*
	1	1	0	15 + 0 = 15*
		0	1	0 + 7 = 7
	2	2	0	18 + 0 = 18
		1	1	15 + 7 = 22*
		0	2	0 + 14 = 14
	3	3	0	
		2	1	
		1	2	
		0	3	
	4	4	0	
		3	1	
		2	2	
		1	3	
		0	4	
	5	5	0	
		4	1	
		3	2	
		2	3	
		1	4	
		0	5	
Joystick Junction	5	5	0	
		4	1	
		3	2	
		2	3	
		1	4	
		0	5	

The Modelling Cycle

Evaluating Solutions

Interpret your mathematical solution(s) in the context of the problem you are modelling.

How accurate and reliable is your solution based on your earlier assumptions?

How can you refine your assumptions to improve your solution and how will this change your solution?

Reflection

What were your key takeaways from this session?

What considerations are needed to take this learning back to your classroom?

Session 3

Exploring Difference Equations

Prior Knowledge

Difference Equations

A Recurrence relation is an equation that defines a sequence where the next term is a function of the previous term(s).

This mathematical relationship often involves the **differences between successive values** of a function of a discrete variable – hence the expression *Difference equations.*

4, 7, 12, 19, 28, 39, ….

0, 1, 3, 14, 57, 227, 966, ….

0, 1, 1, 2, 3, 5, 8, 13, 21 ….

1, 2, 2, 4, 8, 32, 256, ….

Prior Knowledge

Recall - Word Problem from National Seminar 3

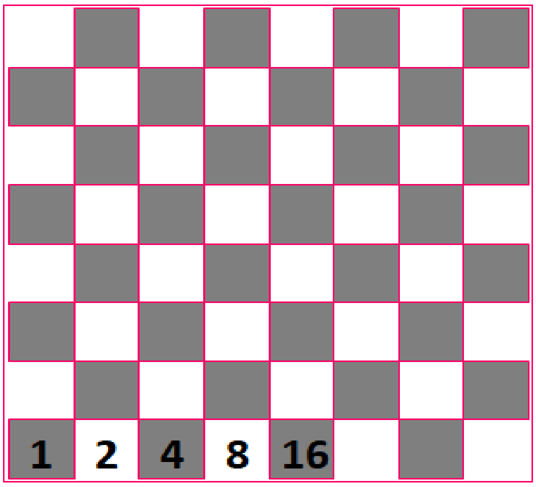
According to legend King Shirham of India wanted to reward his servant for inventing and presenting him with the game of chess. The desire of his servant seemed modest: "Give me a grain of wheat to put on the first square of this chessboard, and two grains to put on the second square, and four grains to put on the third, and eight grains to put on the fourth and so on, doubling for each successive square, give me enough grain to cover all 64 squares."

"You don't ask for much. Your wish will certainly be granted" exclaimed the king.

Based on an extract from "One, Two, Three…Infinity", Dover Publications

Prior Knowledge

Junior Certificate Mathematics



$T_1 = 1$ $T_1 = 1 = 2^0$

$T_2 = 2$ $T_2 = 2 = 2^1$

$T_3 = 4$ $T_3 = 4 = 2^2$

$T_4 = 8$ $T_4 = 8 = 2^3$

$T_5 = 16$ $T_5 = 16 = 2^4$

$S_{64} = 2^0 + 2^1 + 2^2 + 2^3 + 2^4 + \ldots + 2^{63}$

$$2 \times S_{64} = 2^1 + 2^2 + 2^3 + 2^4 + \ldots + 2^{63} + 2^{64}$$

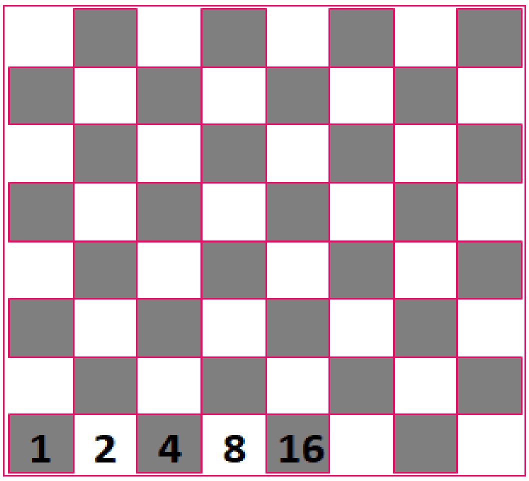
$$- S_{64} = -(2^0 + 2^1 + 2^2 + 2^3 + 2^4 + \ldots + 2^{63})$$

$$= 2^{64} - 2^0 = 2^{64} - 1$$

$$S_{64} = 2^{64} - 1$$

Prior Knowledge

Leaving Certificate Mathematics



$T_1 = 1$ $T_1 = 1 = 2^0$

$T_2 = 2$ $T_2 = 2 = 2^1$

$T_3 = 4$ $T_3 = 4 = 2^2$

$T_4 = 8$ $T_4 = 8 = 2^3$

$T_5 = 16$ $T_5 = 16 = 2^4$

Recurrence relation $T_n = 2^{n-1}$ $n \varepsilon N, n > 1$

$S_{64} = 1 + 2^1 + 2^2 + 2^3 + 2^4 + \ldots + 2^{63}$

This a geometric series

The first term a =1, the ratio r = 2

$$S_n = \frac{a(r^n - 1)}{(r - 1)} \Rightarrow S_{64} = \frac{1(2^{64} - 1)}{(2 - 1)} \quad S_{64} = (2^{64} - 1)$$

What Decides the Order of an Equation?

Consider the sequence of numbers: 0, 1, 1, 2, 3, 5, 8, 13, 21 ….

Recurrence relation	$U_{n+1} = U_n + U_{n-1}$ $n>1,\ n\varepsilon N$

Order = *difference between the iterates*

= *(n+1)-(n-1)*

=> *Order* ***= 2***

This equation is called **homogeneous** because each term is determined by its previous terms only.

Determine the order of the following

Difference Equation	Order of the Equation	Homogeneous or InHomogeneous
$5U_{n+1}+6U_n=0$	1	Homogeneous
$3U_{n+2}+U_{n+1}-2U_n=0$	2	Homogeneous
$U_{n+2}\ -9U_n=0$	2	Homogeneous
$U_{n+3}\ -5U_{n+1}+6=0$	2	InHomogeneous

Characteristic Equation

A Characteristic Equation assists us in determining an expression for **any term** whether we know its preceding terms or not.

Consider the 2nd order difference equation $U_{n+2} - 5U_{n+1} + 6U_n = 0$
We see the coefficients of each term are $\boldsymbol{1}U_{n+2}$ - $\boldsymbol{5}U_{n+1}$+$\boldsymbol{6}U_n$=0

Difference Equation	Homogeneous or InHomogeneous	Characteristic Equation	Roots of Equation
$U_{n+2} - 5U_{n+1} + 6U_n = 0$	Homogeneous	$1X^2 - 5X + 6 = 0$	X=2 , X=3

Group Work

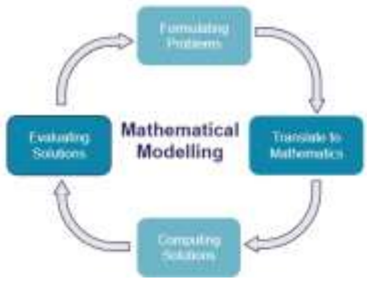
In groups, consider the 2nd Order homogeneous difference equations shown and determine both the characteristic equation and the roots of those equations.

$5U_{n+2} - 6U_n = 0$
$3U_{n+2} + U_{n+1} - 2U_n = 0$
$U_{n+2} - 6U_{n+1} + 9U_n = 0$

Mathematical Modelling Brief

In many real-world settings the management of stock is an important consideration.

Choose a real-world problem related to the distribution or management of stock and model the problem(s) you have selected using *The Modelling Cycle*.



Mathematical Modelling Problem

Problem Statement:

Determine the population of trout in the river Slaney over the next few years, following the introduction of a small number of trout to the river prior to their annual breeding season.

Student-Led Enquiry

In groups,

- discuss what background research that students might consider conducting in order to bring clarity to this problem.
- consider any assumptions students may make.

Outcome of Discussion

At the start of 2021 biologists introduced **twelve trout to** an isolated area of the river just before their **annual breeding season**.

They found that the population had **doubled** by the start of 2022.

The biologists responsible assumed that the current population of trout may be modelled using a difference equation.

Determine the Population of Trout

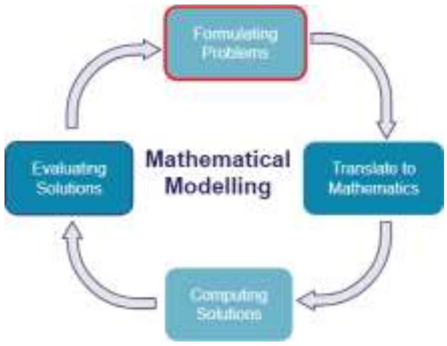
Formulate the problem - Assumptions

The biologist assumed that the current population of trout may be modelled by the following difference equation:

$$P_n = 2.1P_{n-1} - 0.9P_{n-2}$$

where P_n is the current population of trout in the river and $n \in N$.

$P_0 = 12$ in 2021 , $P_1 = 24$ in 2022



Determine the Population of Trout

Translate to Mathematics

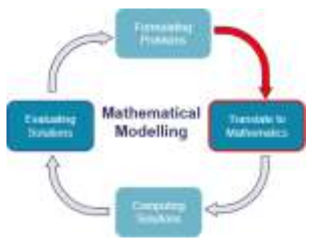
$P_n = 2.7P_{n-1} – 1.8P_{n-2}$ where P_n is the current population of trout in the river and $n \in N$.

What type of equation does this represent?

We have to

(i) Solve this difference equation.

(ii) Calculate the population of trout for say the following two years



Determine the Population of Trout

Computing the Solution

Characteristic equation $x^2 - 2.7x + 1.8 = 0$

Solving this quadratic leads to $x=3/2$ and $x=6/5$

Two distinct roots α , β
$U_n = (l\alpha^n) + (m\beta^n)$

The roots are different therefore we will $P_n = l(\alpha^n) + m(\beta^n)$

$$P_n = l(3/2)^n + m(6/5)^n$$

Taking $P_0=12$ and $P_1=24$ we get

$P_0=l(3/2)^0+m(6/5)^0 = 12 \quad \rightarrow \quad l+m=12$

$P_1=l(3/2)^1+m(6/5)^1 = 24 \quad \rightarrow \quad 5l+4m=80$

Determine the Population of Trout

Computing the Solution

The roots are different we will use equation $P_n = l(\alpha)^n + m(\beta)^n$

Solving these simultaneous equations

$l + m = 12$

$\underline{5l + 4m = 80}$

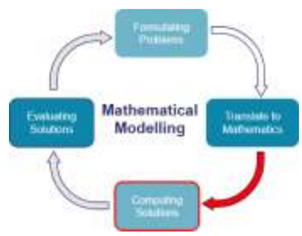
$l = 32$ and $m = -20$

$\Rightarrow P_n=32(3/2)^n - 20(6/5)^n$

This is the solution for the difference equation.

$P_2 = 32(3/2)^2 - 20(6/5)^2 = 43.2 \approx 43$ trout in 2023

$P_3=32(3/2)^3 - 20(6/5)^3 = 73.44 \approx 73$ trout in 2024



Determine the Population of Trout

Evaluating the Solution

12 trout in 2021

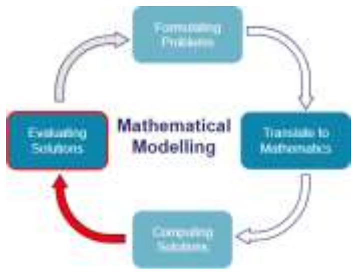
24 trout in 2022

43 trout in 2023

73 trout in 2024

Does this seem accurate based on earlier assumptions?

What effect would changing your variables/assumptions have on your solution?

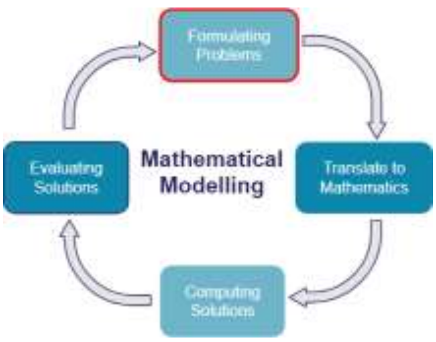


Determine the Population of Trout

Formulate the problem - Assumptions

The biologists, flush with success, adjusted their model to factor in the redistribution of trout to other Irish rivers.

At the start of 2025 Biologists plan to **remove twenty trout** from the Slaney and rehome them in rivers throughout the country.



The biologist revised their model as follows:

$P_n = 2.7P_{n-1} – 1.8P_{n-2} - 20$ where P_n is the current population of trout in the river and $n \in \mathbb{N}$

Determine the Population of Trout

Translate to Mathematics

$P_n = 2.7P_{n-1} – 1.8\ P_{n-2}$ - 20 where P_n is the current population of trout in the river and nN

What type of equation does this represent?

Using this revised model, calculate the revised populations in 2025 and 2026.

Determine the Population of Trout

Computing the Solution

Taking P_0=43 and P_1=73

Population in 2025 $P_n = 2.7(73) – 1.8(43) - 20 = 99.7 \approx$ 99 trout in 2025

Population in 2026 $P_n = 2.7(99) – 1.8(73) - 20 = 115.9 \approx$ 115 trout in 2026

Revised modelling Equation: $P_n = 2.7P_{n-1} – 1.8P_{n-2} – 20$

Rearranging to find particular solution: $P_n - 2.7P_{n-1} + 1.8P_{n-2} = 20$

$$P_n \quad - 2.7P_{n-1} \quad + 1.8P_{n-2} \quad = 0n-20$$

$$(an+b) \quad -2.7(a(n-1)+b) \quad + 1.8(a(n-2)+b) \quad = 0n-20$$

$$an+b \quad -2.7(an-a+b) \quad + 1.8(an-2a+b) \quad = 0n-20$$

$$an+b \quad -2.7an + 2.7a -2.7b + 1.8an -3.6a + 1.8b \quad = 0n-20$$

$$0.1an = 0n \Rightarrow \mathbf{a = 0}$$

$$-0.1b = 20 \Rightarrow \mathbf{b = -200}$$

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F(n)	**Particular solution**
constant	constant a
Kn	an + b
Kn+c	an + b
Kn^2	$an^2 + bn + c$
$Kn^2 + ln + m$	$an^2 + bn + c$
kp^n	$ap^n + b$

A **Second order inhomogeneous equation** is of the form:

$$aP_n + bP_{n-1} + cP_{n-2} = F(n)$$

F(n)	Particular solution
constant	constant a
Kn	an + b
Kn+c	an + b
Kn²	an² + bn + c
Kn² + ln + m	an² + bn + c
kpⁿ	apⁿ + b

The solution to an **inhomogeneous equation** has two components

P_n = [general soln. of associated homogeneous difference equation] + [particular soln. of full equation]

So, to solve an **inhomogeneous difference equation** we must first find the general solution to the associated equation (also known as the complimentary equation) and then the particular solution to the inhomogeneous equation.

Determine the Population of Trout

Computing the Solution

If we use the estimate for 2025 and 2026, we get:

$P_n = l(3/2)^n + m(6/5)^n - 200$

Using earlier estimates *2025 P_0=99* and *2026 P_1=115*

$P_n = l(3/2)^n + m(6/5)^n - 200$

$P_0 = l(3/2)^0 + m(6/5)^0 - 200 = 99 \Rightarrow \quad l + m = 299$

$P_1 = l(3/2)^1 + m(6/5)^1 - 200 = 115 \Rightarrow \quad 15l + 12m = 3150$

Solving these simultaneous equations $l = -146$ and $m = 445$

$$P_n = (-146)\left(\frac{3}{2}\right)^n + (445)\left(\frac{6}{5}\right)^n - 200$$

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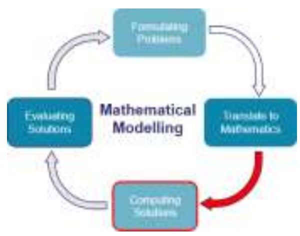
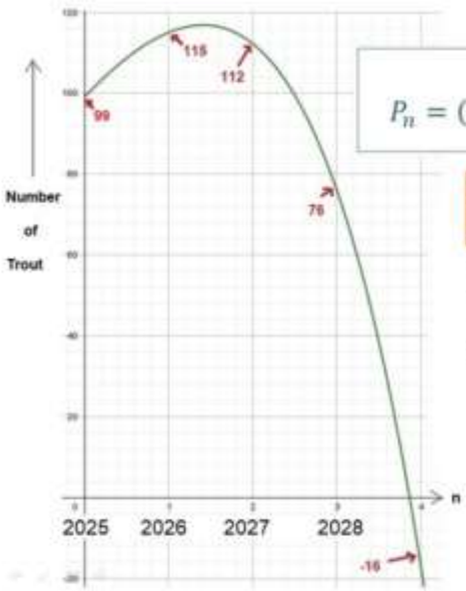


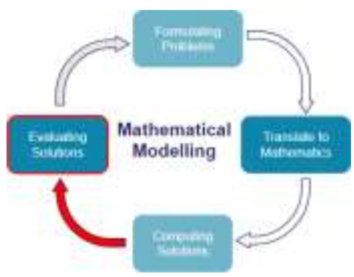
Evaluating the Solution



$$P_n = (-146)\left(\frac{3}{2}\right)^n + (445)\left(\frac{6}{5}\right)^n - 200$$

2025	P_0	$= 99$
2026	P_1	$= 115$
2027	P_2	$= 112$
2028	P_3	$= 76$
2029	P_4	$= -16$

[from earlier estimates]



If we use the P_0=43 (2023) and P_1=73 (2024), we get:

$P_n = l(3/2)^n + m(6/5)^n - 200$

However, if we used *2023 P_0=43* and *2024 P_1=73*

$P_n = l(3/2)^n + m(6/5)^n - 200$

$P_0 = l(3/2)^0 + m(6/5)^0 - 200 = 49 \Rightarrow \quad l + m = 243$

$P_1 = l(3/2)^1 + m(6/5)^1 - 200 = 73 \Rightarrow \quad 15l + 12m = 2730$

Solving these simultaneous equations $l = -62$ and $m = 305$

$$P_n = (-62)\left(\frac{3}{2}\right)^n + (305)\left(\frac{6}{5}\right)^n - 200$$

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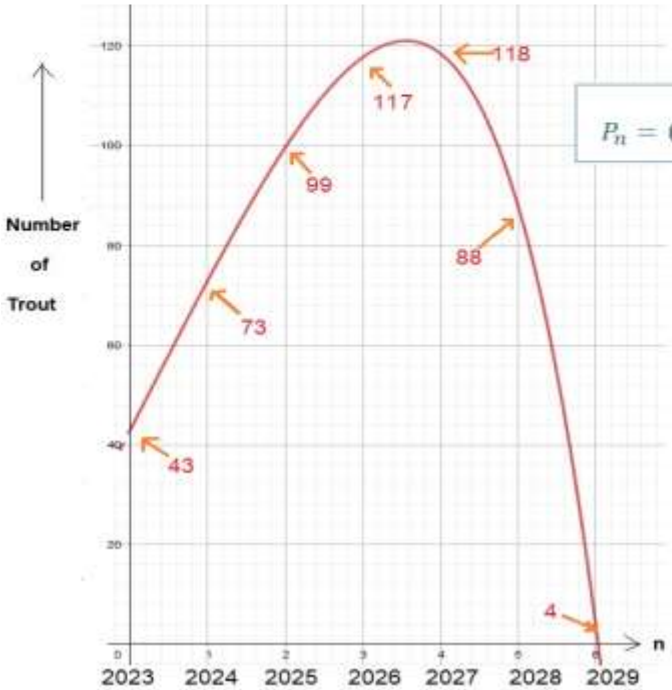
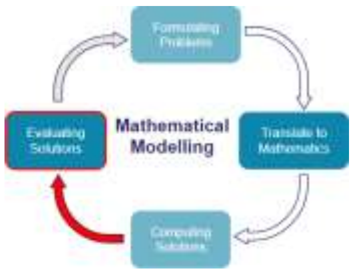


$$P_n = (-62)\left(\frac{3}{2}\right)^n + (305)\left(\frac{6}{5}\right)^n - 200$$

2023	P_0	$= 43$
2024	P_1	$= 73$
2025	P_2	$= 99$
2026	P_3	$= 117$
2027	P_4	$= 118$



Extending The Learning

Original problem

Determine the Population of Trout in river Slaney

"..being able to critically evaluate mathematical models is a desirable skill for them to acquire" p.16

Evaluating the Solution:

How accurate and reliable is your solution based on your earlier assumptions?
What effect would changing your variables/assumptions have on your solution?
How does your solution compare with previous solutions/iterations?
Can you refine/alter your assumptions to improve your solution and will this change your solution much?

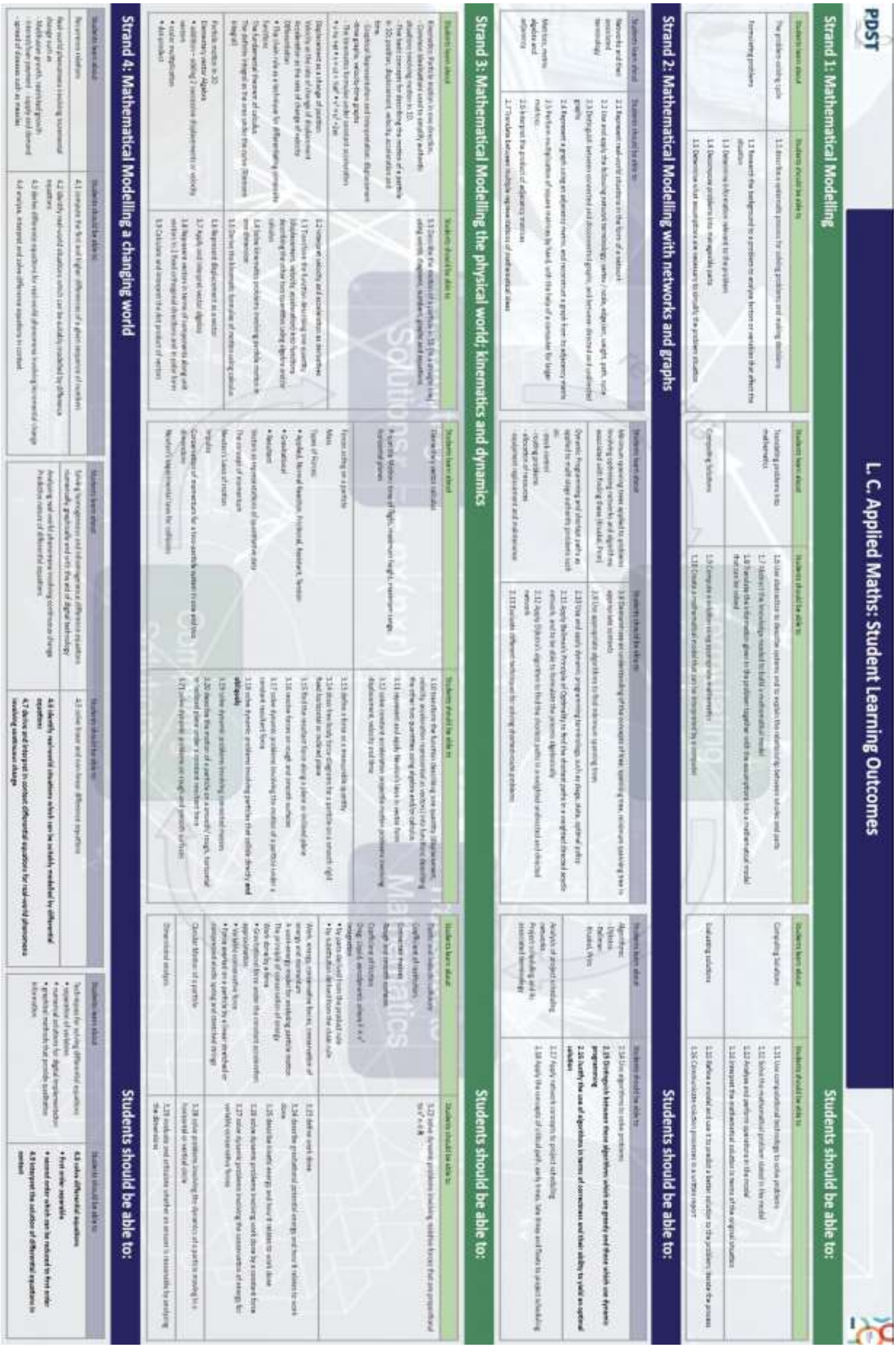
What might a further iteration look like?

How could the model be refined to improve its accuracy?

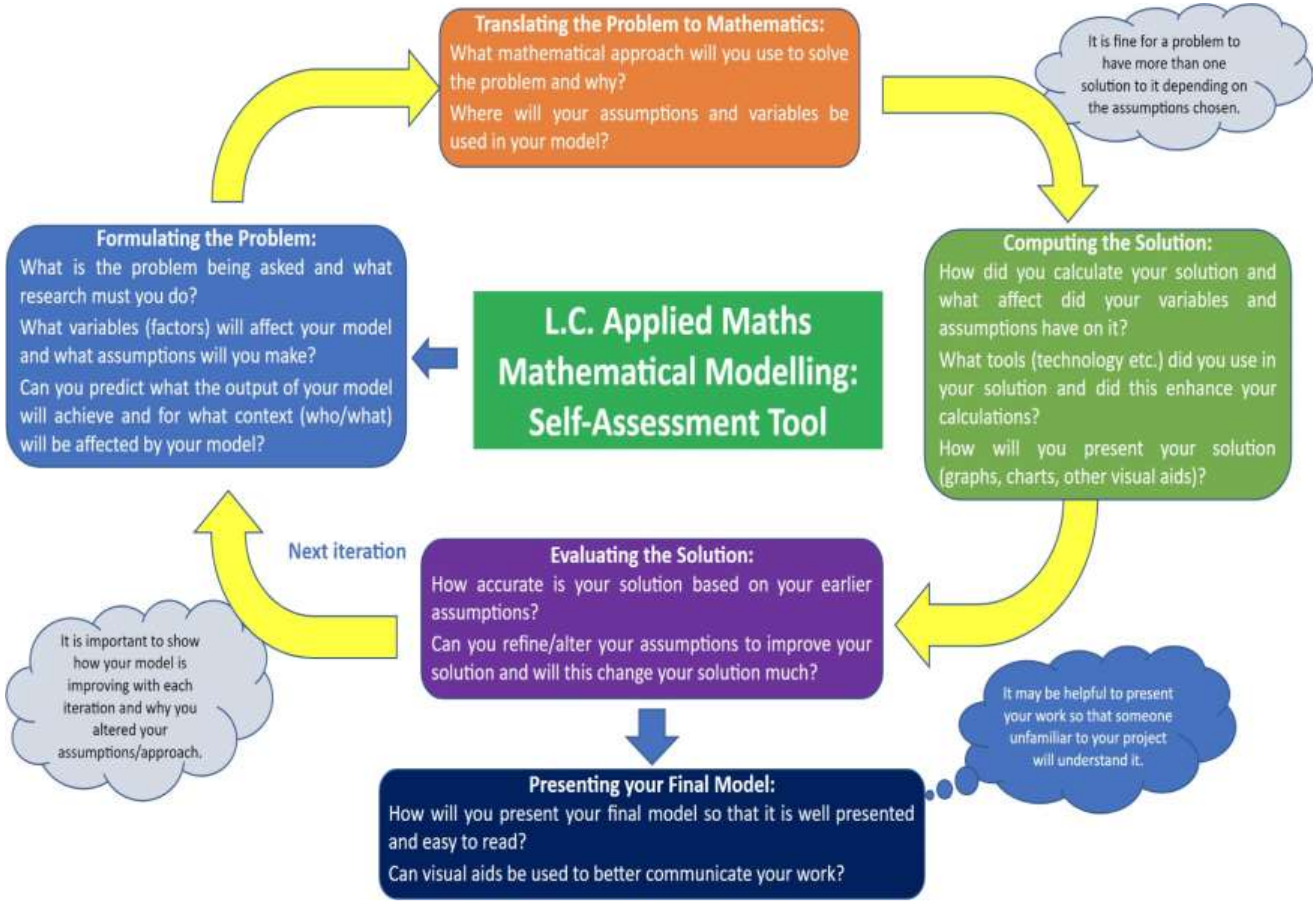
Reflection

How well did this session assist you in your understanding of how difference equations can be developed and formalised through authentic modelling problems?

How useful/relevant did you find today's cross-curricular linking to mathematics?



Oide
Tacú leis an bhFoghlaim Ghairmiúil i measc Ceannairí Scoile agus Múinteoirí
Supporting the Professional Learning of School Leaders and Teachers


L.C. Applied Maths Mathematical Modelling: Self-Assessment Tool
Translating the Problem to Mathematics:
What mathematical approach will you use to solve the problem and why?
Where will your assumptions and variables be used in your model?
It is fine for a problem to have more than one solution to it depending on the assumptions chosen.
Computing the Solution:
How did you calculate your solution and what affect did your variables and assumptions have on it?
What tools (technology etc.) did you use in your solution and did this enhance your calculations?
How will you present your solution (graphs, charts, other visual aids)?
Evaluating the Solution:
How accurate is your solution based on your earlier assumptions?
Can you refine/alter your assumptions to improve your solution and will this change your solution much?
It may be helpful to present your work so that someone unfamiliar to your project will understand it.
Presenting your Final Model:
How will you present your final model so that it is well presented and easy to read?
Can visual aids be used to better communicate your work?
Next iteration
It is important to show how your model is improving with each iteration and why you altered your assumptions/approach.
Formulating the Problem:
What is the problem being asked and what research must you do?
What variables (factors) will affect your model and what assumptions will you make?
Can you predict what the output of your model will achieve and for what context (who/what) will be affected by your model?